Synoptic Meteorology II

Lecture Notes

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Contents

1 A Review on Vector Calculus **1**

2 Rotating Frames and the Momentum Equations **3**

3 Geostrophic, Hydrostatic, and Thermal Wind Balance **9**

4 The Anelastic and Hydrostatic Primitive Equations **13**

5 The Quasi-Geostrophic Approximation **21**

6 Vorticity and Quasi-Geostrophic Vorticity **30**

7 Pressure Coordinates and the Quasi-Geostrophic Geopotential Tendency Equation **35**

8 The Quasi-Geostrophic Omega Equation **43**

9 Q-Vectors and the Ageostrophic Circulation **55**

10 Frontogenesis and Semi-Geostrophy **65**

A Review on Vector Calculus

Chapter 1

Fall 2026: Synoptic Meteorology II

Recall that the derivative is defined as:

$$f'(x) = \frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{1}$$

Suppose we have a property of a fluid, ϕ:

$$\phi(x, y, z)$$

where ϕ is a function of space (but not time). For example, ϕ could be temperature or pressure. Then, we can define the x (or zonal) derivative of ϕ:

$$\frac{\partial \phi}{\partial x} = \lim_{h \to 0} \frac{\phi(x+h, y, z) - \phi(x, y, z)}{h}$$

Note that this is a **partial** derivative because we are traversing in the x-coordinate, while holding our position in the y- and z-coordinates fixed.

Now consider an air parcel moving through the fluid, where the location is specified by (x, y, z). Suppose the position moves in time, such that $x(t)$, $y(t)$, $z(t)$. Then ϕ following the parcel will also be a function of time. How does ϕ following the parcel change over time? We can quantify this using the total derivative, $\frac{d\phi}{dt}$. Note here that the derivative is now "d", because while ϕ is a function of space (i.e. $\phi(x, y, z)$), x, y, and z are functions of time! In the infinitesimal time step described by dt, we define the air parcel moving from:

$$(x_0, y_0, z_0) \ \rightarrow \ (x_0 + \delta x, y_0 + \delta y, z_0 + \delta z)$$

Let us use the chain rule to quantify $\delta\phi$, or the change in ϕ over the time step δt:

$$\delta\phi = \frac{\partial \phi}{\partial t}\delta t + \frac{\partial \phi}{\partial x}\delta x + \frac{\partial \phi}{\partial y}\delta y + \frac{\partial \phi}{\partial z}\delta z \tag{2}$$

$$= \frac{\partial \phi}{\partial t}\delta t + \delta\vec{x} \cdot \nabla\phi \tag{3}$$

where the second step defines $\vec{x}$ as the position vector, and ∇ as the gradient operator:

$$\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \nabla = \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix}$$

It is **very** important to make sure you understand these definitions and are comfortable with using vector calculus to describe physical motions of the atmosphere.

Let us continue with Eq. (3), and divide by δt and take the limit as this time step goes to zero:

$$\begin{aligned}\lim_{\delta t\to 0}\Big[\frac{\delta\phi}{\delta t} &= \frac{\partial\phi}{\partial t}\frac{\delta t}{\delta t} + \frac{\delta\vec{x}\cdot\nabla\phi}{\delta t}\Big]\\ \frac{d\phi}{dt} &= \frac{\partial\phi}{\partial t} + \lim_{\delta t\to 0}\Big[\frac{\delta x}{\delta t}\frac{\partial\phi}{\partial x} + \frac{\delta y}{\delta t}\frac{\partial\phi}{\partial y} + \frac{\delta z}{\delta t}\frac{\partial\phi}{\partial z}\Big]\\ &= \frac{\partial\phi}{\partial t} + \frac{dx}{dt}\frac{\partial\phi}{\partial x} + \frac{dy}{dt}\frac{\partial\phi}{\partial y} + \frac{dz}{dt}\frac{\partial\phi}{\partial z}\end{aligned}$$

Then, if we define the zonal (west-east) velocity u, the meridional (south-north) velocity v, and vertical (down-up) velocity w:

$$u \equiv \frac{dx}{dt}, \qquad v \equiv \frac{dy}{dt}, \qquad w \equiv \frac{dz}{dt}$$

and the velocity vector as $\vec{u}$

$$\vec{u} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \frac{d\vec{x}}{dt}$$

the above simplifies to:

$$\frac{d\phi}{dt} = \frac{D\phi}{Dt} = \frac{\partial\phi}{\partial t} + (\vec{u}\cdot\nabla)\phi \tag{4}$$

This is a very important equation that describes the change in ϕ following a parcel. Notice we have given a special notation for the total derivative of ϕ with respect to time, by replacing "d" with "D". The total derivative is given a special name, the material derivative. It specifically describes the effect of **advection** on the change of a fluid property over time.

For example, consider T as the property of the fluid.

$$\frac{DT}{Dt} = \underbrace{\frac{\partial T}{\partial t}}_{\text{local}} + \underbrace{(\vec{u}\cdot\nabla)T}_{\text{advection}} \tag{5}$$

Finally, another useful operator that we will use a lot in this course is the Laplacian operator:

$$\nabla^2 = \nabla\cdot\nabla \tag{6}$$

and applying it to our fluid property ϕ yields:

$$\nabla^2\phi = \Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\Big)\phi \tag{7}$$

Rotating Frames and the Momentum Equations

Chapter 2

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Rotating Frames

Motion in a Rotating Frame (see Vallis 2.1, Holton 3.1 for more)

Consider a vector $\vec{C}$ rotating relative to an inertial frame with angular velocity $\vec{\Omega}$. $\vec{C}$ is rotating rigidly about the axis $\vec{\Omega}$, like a spoke attached to a spinning wheel. As it rotates, the tip of $\vec{C}$ traces out a circle around the $\vec{\Omega}$ axis.

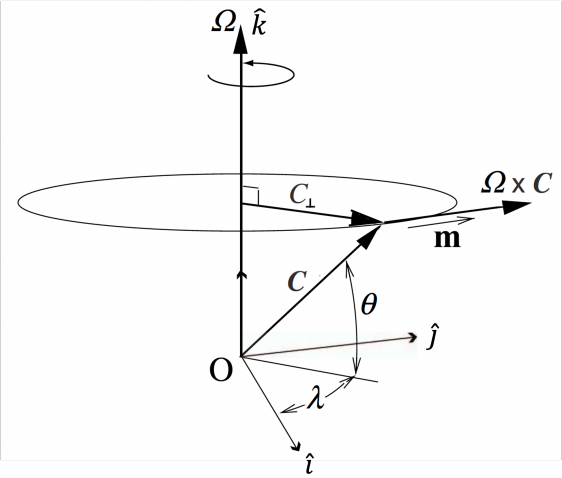


Figure 1: Vector $\vec{C}$ rotating about $\vec{\Omega}$ (Vallis 2.1).

- $\lambda \rightarrow$ angle traveled in the $\hat{\imath}$–$\hat{\jmath}$ plane
- $\theta \rightarrow$ angle between $\vec{C}$ and the $\hat{\imath}$–$\hat{\jmath}$ plane
- $\hat{m} \rightarrow$ unit vector parallel to the direction of change of $\vec{C}$

Next, consider the displacement vector of $\vec{C}$, which is [how far you moved] multiplied by [which direction you moved]. Thus, the (infinitesimal) change in $\vec{C}$ is

$$\delta\vec{C} = (\text{arc length change})\,\hat{m} = (\text{radius}) \cdot (\text{rate of change of angle}) \cdot \delta t\,\hat{m} = |\vec{C}|\cos\theta\,\delta\lambda\,\hat{m}.$$

Since $\delta\lambda = |\vec{\Omega}|\,\delta t$,

$$\delta\vec{C} = |\vec{C}|\cos\theta\,|\vec{\Omega}|\,\delta t\,\hat{m}.$$

Trig identity: $\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$

Applying the identity with $a = \frac{\pi}{2}$ and $b = -\theta$,

$$\sin\left(\tfrac{\pi}{2} - \theta\right) = \sin\left(\tfrac{\pi}{2}\right)\cos(-\theta) + \cos\left(\tfrac{\pi}{2}\right)\sin(-\theta) = (1)\cos\theta + (0)(-\sin\theta) = \cos\theta,$$

so that $\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$. Substituting this in,

$$\delta\vec{C} = |\vec{C}|\,\sin\left(\tfrac{\pi}{2} - \theta\right)|\vec{\Omega}|\,\delta t\,\hat{m} = \vec{\Omega} \times \vec{C}\,\delta t,$$

recalling the definition of the cross product, $\vec{a} \times \vec{b} \equiv |\vec{a}||\vec{b}|\sin\hat{\theta}\,\hat{n}$ (with $\vec{a} = \vec{C}$, $\vec{b} = \vec{\Omega}$), where $\hat{\theta}$ is the angle between $\vec{a}$ and $\vec{b}$. Therefore,

$$\boxed{\left(\frac{d\vec{C}}{dt}\right)_{\text{inertial}} = \vec{\Omega} \times \vec{C}} \qquad \text{(Eq. 1)}$$

(What is the dot product of the RHS with $\vec{C}$?)

Now consider a vector $\vec{V}$ that changes in the inertial frame,

$$\vec{V} = V_1\,\hat{\imath}_R + V_2\,\hat{\jmath}_R + V_3\,\hat{k}_R, \qquad (\hat{\imath}, \hat{\jmath}, \hat{k})_R \text{ are unit vectors of the rotating frame.}$$

The rate of change of $\vec{V}$ in the inertial frame is

$$\left(\frac{d\vec{V}}{dt}\right)_I = \left(\frac{dV_1}{dt}\right)_I \hat{\imath}_R + V_1\left(\frac{d\hat{\imath}_R}{dt}\right)_I + \left(\frac{dV_2}{dt}\right)_I \hat{\jmath}_R + V_2\left(\frac{d\hat{\jmath}_R}{dt}\right)_I + \left(\frac{dV_3}{dt}\right)_I \hat{k}_R + V_3\left(\frac{d\hat{k}_R}{dt}\right)_I$$

Since V_1, V_2, V_3 are scalars, $\left(\frac{dV_1}{dt}\right)_I = \left(\frac{dV_1}{dt}\right)_R$, and using Eq. 1 on each rotating unit vector,

$$\left(\frac{d\vec{V}}{dt}\right)_I = \left(\frac{dV_1}{dt}\right)_R \hat{\imath}_R + V_1(\vec{\Omega} \times \hat{\imath}_R) + \left(\frac{dV_2}{dt}\right)_R \hat{\jmath}_R + V_2(\vec{\Omega} \times \hat{\jmath}_R) + \cdots$$

Gathering terms:

$$\boxed{\left(\frac{d\vec{V}}{dt}\right)_I = \left(\frac{d\vec{V}}{dt}\right)_R + \vec{\Omega} \times \vec{V}} \qquad \text{(Eq. 2)}$$

This relates vector changes in the inertial frame to the rotating frame!

Relating Position and Velocity in the Inertial and Reference Frames

Apply Eq. 2 to the position vector $\vec{r}$:

$$\left(\frac{d\vec{r}}{dt}\right)_I = \left(\frac{d\vec{r}}{dt}\right)_R + \vec{\Omega} \times \vec{r} \quad \Longrightarrow \quad \vec{V}_I = \vec{V}_R + \vec{\Omega} \times \vec{r},$$

i.e. the *inertial velocity* equals the *measured (relative) velocity* plus the rotating frame term $\vec{\Omega} \times \vec{r}$.

Apply Eq. 2 again, now to the relative velocity $\vec{V}_R$:

$$\left(\frac{d\vec{V}_R}{dt}\right)_I = \left(\frac{d\vec{V}_R}{dt}\right)_R + \vec{\Omega} \times \vec{V}_R.$$

Since $\vec{V}_R = \vec{V}_I - \vec{\Omega} \times \vec{r}$,

$$\left(\frac{d(\vec{V}_I - \vec{\Omega} \times \vec{r})}{dt}\right)_I = \left(\frac{d\vec{V}_R}{dt}\right)_R + \vec{\Omega} \times \vec{V}_R.$$

Expanding by the chain rule (with $d\vec{\Omega}/dt = 0$) and using $\left(\frac{d\vec{r}}{dt}\right)_I = \left(\frac{d\vec{r}}{dt}\right)_R + \vec{\Omega} \times \vec{r} = \vec{V}_R + \vec{\Omega} \times \vec{r}$:

$$\left(\frac{d\vec{V}_I}{dt}\right)_I - \vec{\Omega} \times \left(\vec{V}_R + \vec{\Omega} \times \vec{r}\right) = \left(\frac{d\vec{V}_R}{dt}\right)_R + \vec{\Omega} \times \vec{V}_R,$$

so that

$$\left(\frac{d\vec{V}_I}{dt}\right)_I - \vec{\Omega} \times (\vec{\Omega} \times \vec{r}) - 2\,\vec{\Omega} \times \vec{V}_R = \left(\frac{d\vec{V}_R}{dt}\right)_R$$

$$\underbrace{\left(\frac{d\vec{V}_I}{dt}\right)_I}_{\substack{\text{accleration in inertial frame} \\ = \vec{F}/m \text{ (Newton's 2nd law)}}} - \underbrace{\vec{\Omega} \times (\vec{\Omega} \times \vec{r})}_{\text{centrifugal force}} - \underbrace{2\,\vec{\Omega} \times \vec{V}_R}_{\text{Coriolis force}} = \underbrace{\left(\frac{d\vec{V}_R}{dt}\right)_R}_{\substack{\text{acceleration in rotating frame} \\ \text{(what we measure)}}}$$

Centrifugal and Coriolis forces are "fictitious" forces to make velocity in the rotating frame of reference, "look" like velocity in the intertial frame of reference!!

Coriolis force $= -2\,\vec{\Omega} \times \vec{V}_R$

1) is zero on stationary bodies;
2) deflects bodies at right angles;
3) does no work, since $\vec{V}_R \cdot (-2\vec{\Omega} \times \vec{V}_R) = 0$.

Centrifugal term: writing $\vec{\Omega} \times \vec{r} = \vec{\Omega} \times \vec{r}_\perp$, where $\vec{r}_\perp$ is the perpendicular distance to the axis of rotation,

$$\vec{\Omega} \times (\vec{\Omega} \times \vec{r}_\perp) = \underbrace{(\vec{\Omega} \cdot \vec{r}_\perp)}_{=0}\,\vec{\Omega} - (\vec{\Omega} \cdot \vec{\Omega})\,\vec{r}_\perp = -\Omega^2 \vec{r}_\perp.$$

The centrifugal force acts a little like gravity, but in the opposite direction, so we incorporate it into an **"effective gravity."** (more on this later...).

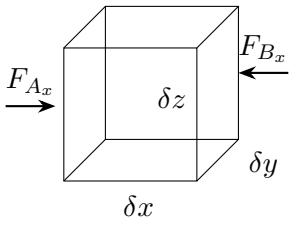
The Momentum Equations

Now we can use Newton's Second Law:

$$\left(\frac{d\vec{V}_I}{dt}\right)_I = \frac{\vec{F}}{m}.$$

to derive an equation that tells us how (atmospheric) motion changes in time. Clearly, this equation tells us that we need to know what are the forces acting on the atmosphere. For now, let's consider *pressure* and *gravity*.

The Pressure Gradient Force



For a small fluid parallelepiped of sides $\delta x, \delta y, \delta z$, the pressure forces on the two faces normal to x are:

$$F_{A_x} = p_0\,\delta y\,\delta z, \qquad F_{B_x} = -\left(p_0 + \frac{\partial p}{\partial x}\delta x\right)\delta y\,\delta z,$$

so the net force is

$$F_x = F_{A_x} + F_{B_x} = -\,\delta x\,\delta y\,\delta z\,\frac{\partial p}{\partial x}.$$

With density $\rho = m/\delta V$, i.e. $m = \rho\,\delta x\,\delta y\,\delta z$,

$$\frac{F_x}{m} = -\frac{1}{\rho}\frac{\partial p}{\partial x}.$$

By similarity,

$$\frac{F_y}{m} = -\frac{1}{\rho}\frac{\partial p}{\partial y}, \qquad \frac{F_z}{m} = -\frac{1}{\rho}\frac{\partial p}{\partial z}.$$

Pressure gradient force: $-\frac{1}{\rho}\nabla p$

Gravity

The other force we include for now is gravity. Unlike the pressure gradient force, gravity acts *only* in the vertical: it pulls fluid parcels straight down, toward the center of the Earth, with no horizontal component. Per unit mass this is just the gravitational acceleration g, so

$$\frac{F_x}{m} = 0, \qquad \frac{F_y}{m} = 0, \qquad \frac{F_z}{m} = -g,$$

where $\hat{k}$ points vertically upward and $g \approx 9.81$ m/s^2. Equivalently, gravity is minus the gradient of a geopotential $\Phi = gz$ (equal to $g\hat{k}$ here after absorbing the $\hat{z}$ direction),

$$-\nabla\Phi = -g\hat{k},$$

so that Φ plays the same role for gravity that p/ρ plays for the pressure gradient force. (Recall from earlier that the centrifugal force can also be written as the gradient of a potential and folded into an *effective* gravity/geopotential.)

Momentum Equations on the f-Plane

Let's write the momentum equations on the f-plane (rotating frame of reference, fixed):

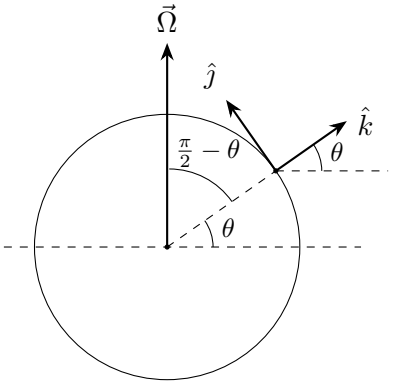
$$\frac{D\vec{v}}{Dt} + 2\vec{\Omega} \times \vec{v} = -\frac{1}{\rho}\nabla p - \nabla\Phi,$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{v} \cdot \nabla, \qquad \Phi = g\hat{k},$$

$$\vec{v} = [u, v, w], \qquad \vec{\Omega} = [\Omega_x, \Omega_y, \Omega_z] = [0,\ \Omega_0 \cos\theta,\ \Omega_0 \sin\theta],$$

and Ω_0 is the angular velocity of the Earth, $\Omega_0 = \dfrac{2\pi \text{ radians}}{1 \text{ day rotation}}$.



Decomposing $\vec{\Omega}$ along the local, surface-fixed axes $(\hat{j}, \hat{k})$ at latitude θ (i.e. $\hat{k}$ points locally "up" and $\hat{j}$ points locally "north"), and using that $\vec{\Omega}$ makes angle $\left(\frac{\pi}{2} - \theta\right)$ with $\hat{k}$ and angle θ with $\hat{j}$,

$$\Omega_k = \Omega_0 \cos\left(\tfrac{\pi}{2} - \theta\right) = \Omega_0 \sin\theta, \qquad \Omega_j = \Omega_0 \cos\theta.$$

At any point on the surface, the local axes are $\hat{\imath}$ (east), $\hat{\jmath}$ (north), and $\hat{k}$ (local vertical, "up"). Since $\vec{\Omega}$ points along the Earth's polar axis, it lies entirely within the local meridian plane – the vertical north–south plane through that point – which is spanned by $\hat{\jmath}$ and $\hat{k}$. The local east direction $\hat{\imath}$ is, by construction, perpendicular to this plane, so $\vec{\Omega}$ has zero projection onto $\hat{\imath}$ at every latitude:

$$\Omega_i = 0.$$

Combined with the components found above, this gives

$$\vec{\Omega} = [\Omega_i, \Omega_j, \Omega_k] = [0,\ \Omega_0 \cos\theta,\ \Omega_0 \sin\theta].$$

As a sanity check: at the pole ($\theta = 90°$), $\vec{\Omega}$ points straight up, giving $\Omega_k = \Omega_0$, $\Omega_j = 0$; at the equator ($\theta = 0°$), $\vec{\Omega}$ points due north along the surface, giving $\Omega_k = 0$, $\Omega_j = \Omega_0$.

Computing the Coriolis term:

$$2\vec{\Omega}\times\vec{v}=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ 0 & 2\Omega_0\cos\theta & 2\Omega_0\sin\theta\\ u & v & w\end{vmatrix}=\big(2\Omega_0 w\cos\theta-2\Omega_0 v\sin\theta\big)\hat{\imath}+\big(2\Omega_0 u\sin\theta\big)\hat{\jmath}-\big(2\Omega_0 u\cos\theta\big)\hat{k}.$$

The Coriolis terms involving w and $\hat{k}$ are usually ignored (thin-shell / traditional approximation), leaving the **Coriolis parameter**

$$\boxed{f=2\Omega_0\sin\theta}$$

Writing out each component of $\dfrac{D\vec{v}}{Dt}+2\vec{\Omega}\times\vec{v}=-\dfrac{1}{\rho}\nabla p-\nabla\Phi$:

$$x:\quad \frac{\partial u}{\partial t}+\vec{v}\cdot\nabla u-2\Omega_0\sin\theta\, v=-\frac{1}{\rho}\frac{\partial p}{\partial x}\quad\Longrightarrow\quad\boxed{\frac{Du}{Dt}-fv=-\frac{1}{\rho}\frac{\partial p}{\partial x}}$$

$$y:\quad \frac{\partial v}{\partial t}+\vec{v}\cdot\nabla v+2\Omega_0\sin\theta\, u=-\frac{1}{\rho}\frac{\partial p}{\partial y}\quad\Longrightarrow\quad\boxed{\frac{Dv}{Dt}+fu=-\frac{1}{\rho}\frac{\partial p}{\partial y}}$$

$$z:\quad \frac{\partial w}{\partial t}+\vec{v}\cdot\nabla w=-\frac{1}{\rho}\frac{\partial p}{\partial z}-g\quad\Longrightarrow\quad\boxed{\frac{Dw}{Dt}=-\frac{1}{\rho}\frac{\partial p}{\partial z}-g}$$

These are the momentum equations!

The x- and y-equations are the **horizontal momentum equations**: they describe motion in the plane tangent to the Earth's surface, and each is balanced by the Coriolis term ($-fv$ or $+fu$) coupling the two horizontal velocity components together. This Coriolis coupling is the key new ingredient introduced by working in a rotating frame — it is absent in the z-equation because we dropped the Coriolis terms involving w and $\hat{k}$ under the traditional approximation. The z-equation is instead the **vertical momentum equation**. Together, these three equations, two horizontal, one vertical, form the momentum equations we will use to describe large-scale, rotating fluid motion on the f-plane.

Geostrophic, Hydrostatic, and Thermal Wind Balance

Chapter 3

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Dimensional Analysis (Vallis 2.7–2.8)

The momentum equations contain many terms, but in any given physical regime only a few of them actually matter. *Dimensional analysis* (or *scale analysis*) is the procedure for finding out which terms are important in certain physical regimes. Let us do a scale analysis for synoptic scale processes.

Variable	Description	Scaling	Real Atmos.
x, y	horizontal scale	L	10^6 m
t	time	T	1 day (10^5 s)
u, v	horizontal velocity	U, V	10 m/s

$$T \sim \frac{L}{U}$$

Next, we'll apply these to the momentum equations that we derived previously, to try to see if any terms simplify.

Horizontal Momentum Equations: Geostrophic Balance

First, let's look at the horizontal momentum equations.

$$\left.\begin{aligned} \frac{Du}{Dt} - fv &= -\frac{1}{\rho}\frac{\partial p}{\partial x} \\ \frac{Dv}{Dt} + fu &= -\frac{1}{\rho}\frac{\partial p}{\partial y} \end{aligned}\right\} \quad \text{horizontal momentum equations}$$

Scale Analysis

$$\begin{array}{ccccccc} \dfrac{\partial u}{\partial t} & + & (\vec{u}\cdot\nabla)u & - & fv & = & -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x} \\ U\dfrac{U}{L} & & U\dfrac{1}{L}U & & fU & & \dfrac{P}{\rho L} \\ \dfrac{U^2}{L} & & \dfrac{U^2}{L} & & fU & & \dfrac{P}{\rho L} \end{array}$$

$u, v : U$ velocity

$t : L/U$ time for fluid to travel characteristic length

$f : f$ coriolis parameter

Ratio of the first two terms with the third term on the right-hand-side:

$$\frac{U^2/L}{fU} = \frac{U}{fL} = \mathrm{Ro} \qquad , \qquad \boxed{\text{Rossby Number}}$$

The Rossby number tells you how important the advective time scale is to the rotational time scale.

- When $\mathrm{Ro} \ll 1$, rotation is important, and the fU term dominates on the RHS
- When $\mathrm{Ro} \gg 1$, advection is important, and the $\dfrac{\partial u}{\partial t}$, $(\vec{u} \cdot \nabla)u$ terms dominate on the RHS

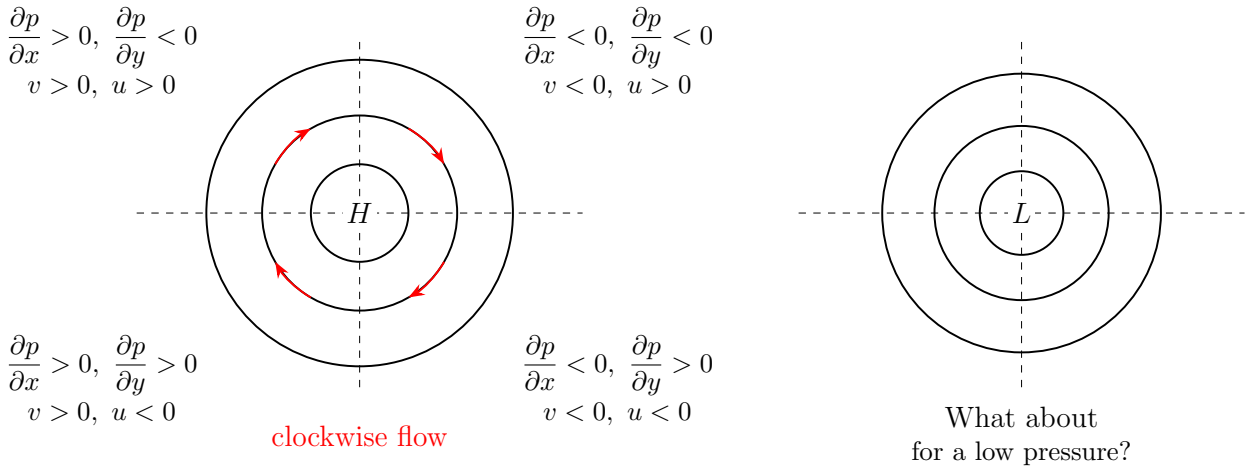
Let's evaluate Ro for synoptic-scale flow using the scales from the table above. We have $U \sim 10$ m/s and $L \sim 10^6$ m, together with a mid-latitude Coriolis parameter of $f \sim 10^{-4}$ s^{-1}.

$$\mathrm{Ro} = \frac{U}{fL} \sim \frac{10 \text{ m/s}}{(10^{-4} \text{ s}^{-1})(10^6 \text{ m})} \sim 0.1$$

Since $\mathrm{Ro} \sim 0.1 \ll 1$ for synoptic-scale motion, rotation dominates over advection, and the fU (Coriolis) term is much larger than the U^2/L (advective) term in the horizontal momentum equations. This means the advective term can be dropped! Therefore,

$$\boxed{fu_g \approx -\frac{1}{\rho}\frac{\partial p}{\partial y} \quad , \quad fv_g \sim \frac{1}{\rho}\frac{\partial p}{\partial x}}$$

where u_g and v_g are the geostrophic velocities. Notice how this equation tells us that when the Rossby number is small, there is a simple balance between pressure gradients and velocity. This is otherwise known as geostrophic balance.



Divergence (on an f-plane)

An important characteristic of geostrophic velocities is that they have no horizontal divergence (or convergence). We can see this by taking the horizontal divergence of $\vec{u}_g$:

$$\nabla_H \cdot \vec{u}_g = \frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y} = -\frac{1}{\rho f}\frac{\partial^2 p}{\partial x \partial y} + \frac{1}{\rho f}\frac{\partial^2 p}{\partial x \partial y} = 0$$

$$\nabla_H \cdot \vec{u}_g = 0$$

Geostrophic flow is horizontally non-divergent.

Vertical Momentum: Hydrostatic Balance

Next, let us visit the momentum equation in the vertical (up and down) direction. Previously, we derived:

$$\frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g$$

Let us now try to scale the vertical momentum equation. Since g is a constant, we will need to scale the advection term. Now, if one assumes the magnitude of w for synoptic scale storms is $W \sim 1$ cm/s (is this a reasonable assumption? we shall see later...), one will find that the terms on the right-hand-side of the vertical momentum terms dominate (this exercise is left to the reader). In this case, the balance must be between the pressure gradient and gravity terms:

$$\boxed{\frac{\partial p}{\partial z} = -\rho g}$$

In other words, the upward pressure gradient force (if pressure decreases with height), balances the downwards gravity force. This balance is called hydrostatic balance.

Combining It All: Thermal Wind

What happens when we combine the joint constraints of geostrophic and hydrostatic balance?

$$-fv_g = -\frac{1}{\rho}\frac{\partial p}{\partial x}\bigg|_{y,z} \qquad \frac{\partial p}{\partial z} = -\rho g$$
$$fu_g = -\frac{1}{\rho}\frac{\partial p}{\partial y}\bigg|_{x,z}$$

In order to simplify this system, we need to change the vertical coordinate. Let us define the geopotential height as $\Phi = gz$. Then, we can rewrite hydrostatic balance (α is specific volume):

$$-\frac{\partial p}{\rho} = g\,dz = \partial\Phi \quad \Longrightarrow \quad \frac{\partial \Phi}{\partial p} = -\frac{1}{\rho} = -\alpha$$

Consider a general vertical coordinate ξ (chain rule):

$$\left(\frac{\partial}{\partial x}\right)_\xi = \left(\frac{\partial}{\partial x}\right)_z + \left(\frac{\partial z}{\partial x}\right)_\xi \frac{\partial}{\partial z}$$

Let $\xi = p$, so

$$\left(\frac{\partial p}{\partial x}\right)_p = \left(\frac{\partial p}{\partial x}\right)_z + \left(\frac{\partial z}{\partial x}\right)_p \frac{\partial p}{\partial z}$$
$$0 = \left(\frac{\partial p}{\partial x}\right)_z - \left(\frac{\partial z}{\partial x}\right)_p \rho g$$
$$= \left(\frac{\partial p}{\partial x}\right)_z - \rho\left(\frac{\partial \Phi}{\partial x}\right)_p$$

$$\boxed{\frac{\partial \Phi}{\partial x}\bigg|_p = \frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right)_z}$$

Therefore, in pressure coordinates, geostrophic and hydrostatic balance simplify to:

$$\boxed{\begin{aligned} -fv_g &= -\left.\frac{\partial \Phi}{\partial x}\right|_p \\ fu_g &= -\left.\frac{\partial \Phi}{\partial y}\right|_p \end{aligned}} \qquad \text{plus} \qquad \frac{\partial \Phi}{\partial p} = -\alpha$$

Taking the pressure derivative $\partial/\partial p$, and using hydrostatic balance:

$$-f\frac{\partial v_g}{\partial p} = -\frac{\partial}{\partial x}\frac{\partial \Phi}{\partial p} = \frac{\partial \alpha}{\partial x} \tag{1}$$

$$f\frac{\partial u_g}{\partial p} = -\frac{\partial}{\partial y}\frac{\partial \Phi}{\partial p} = \frac{\partial \alpha}{\partial y} \tag{2}$$

We could stop here, but α is not as easy to think about as compared to temperature. So we will simply substitute α for T, using the ideal gas equation. The ideal gas equation can be written as:

$$\boxed{\rho = \frac{p}{R^* T}}$$

where $\dfrac{R}{M} = R^*$ is the specific gas constant (equal to 287 $\frac{\text{J}}{\text{kg} \cdot \text{mol}}$). Then, since $\rho = 1/\alpha$,

$$\boxed{\frac{\partial \Phi}{\partial p} = -\frac{R^* T}{p}}$$

Substituting the above into the Eqns. 1 and 2:

$$-f\frac{\partial v_g}{\partial p} = -\frac{\partial}{\partial x}\left(\frac{R^* T}{p}\right) = -\frac{R^*}{p}\frac{\partial T}{\partial x}$$

$$f\frac{\partial u_g}{\partial p} = -\frac{\partial}{\partial y}\left(\frac{R^* T}{p}\right) = -\frac{R^*}{p}\frac{\partial T}{\partial y}$$

which finally yields the thermal wind balance equations:

Thermal Wind Balance

$$\boxed{\begin{aligned} \frac{\partial v_g}{\partial p} &= -\frac{R^*}{fp}\frac{\partial T}{\partial x} \\ \frac{\partial u_g}{\partial p} &= \frac{R^*}{fp}\frac{\partial T}{\partial y} \end{aligned}}$$

Clearly, this equation relates the change in pressure (or height) for the geostrophic wind, with the horizontal temperature gradient (on constant pressure surfaces). In other words **horizontal temperature gradients results in vertical wind shear**! This is the important equation in this entire course, and we will spend a lot of time both trying to understand it, and using it.

The Anelastic and Hydrostatic Primitive Equations

Chapter 4

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Revisiting Geostrophic Balance

Recall that geostrophic balance can be written compactly as

$$\vec{u}_g = \frac{1}{\rho f}\,\hat{k} \times \nabla_H p \qquad \Longleftrightarrow \qquad \left.\begin{aligned} f v_g &= \frac{1}{\rho}\frac{\partial p}{\partial x} \\ -f u_g &= \frac{1}{\rho}\frac{\partial p}{\partial y} \end{aligned}\right\} \quad \text{back in } z\text{-coordinates}$$

Geostrophic balance is only a balance! It only tells you which terms are balancing each other, much like a elevator counterweight balances the weight of the elevator car. The same is true of hydrostatic balance and thermal wind balance. **They do not tell you how the atmosphere changes in time!** (Notice how there is no $\frac{D}{Dt}$ term in the geostrophic balance equation).

Recall we derived in class:

$$\frac{D\vec{u}}{Dt} + 2\vec{\Omega} \times \vec{u} = -\frac{1}{\rho}\nabla p - \nabla\Phi$$

where we threw out the material derivative term for $\mathrm{Ro} \ll 1$. However, that term is not totally negligible and equal to zero, it's just small! Therefore, for small Ro, there must be a correspondingly small, but non-negligible component of the wind that is not in geostrophic balance.

Let us try to quantify this by decomposing the full velocity into

$$\vec{u} = \vec{u}_g + \vec{u}_a$$

where $\vec{u}_a$ is the ageostrophic wind (or the component of wind that is not in geostrophic balance). Note that $\vec{u}_a$ is much smaller than $\vec{u}_g$.

Retaining the Ageostrophic Wind

Now that we have decomposed the wind into its geostrophic and ageostrophic components, let us try to rewrite the full horizontal momentum equations, recalling geostrophic balance first:

$$f v_g = \frac{1}{\rho}\frac{\partial p}{\partial x} \tag{1}$$

$$f u_g = -\frac{1}{\rho}\frac{\partial p}{\partial y} \tag{2}$$

Then,

$$\begin{aligned} \frac{Du}{Dt} - fv &= -\frac{1}{\rho}\frac{\partial p}{\partial x} \\ \frac{Dv}{Dt} + fu &= -\frac{1}{\rho}\frac{\partial p}{\partial y} \end{aligned} \quad \Longrightarrow \quad \begin{aligned} \frac{Du}{Dt} - fv &= -fv_g \\ \frac{Dv}{Dt} + fu &= fu_g \end{aligned} \quad \Longrightarrow \quad \begin{aligned} \frac{Du}{Dt} &= f(v - v_g) \\ \frac{Dv}{Dt} &= -f(u - u_g) \end{aligned}$$

Thus, using $\vec{u}_a = \vec{u} - \vec{u}_g$,

$$\frac{Du}{Dt} = fv_a$$

$$\frac{Dv}{Dt} = -fu_a$$

or,

$$\boxed{\frac{D\vec{u}}{Dt} = -f\,\hat{k} \times \vec{u}_a} \tag{3}$$

In other words, **the evolution of $\vec{u}$, the total atmospheric flow, is directly linked to the ageostrophic wind, not the geostrophic wind**! This is a very important concept to understand. Geostrophic balance tells us nothing about how the atmosphere changes in time. And, in fact, it is the part of the atmosphere that is **not** in geostrophic balance (the ageostrophic component) that determines how the atmosphere evolves in time.

Scale Analysis

Let us scale Eq. 3 above:

$$\frac{U^2}{L} \sim fU_a$$

where U_a is the magnitude of the ageostrophic wind. Dividing by fU gives us the well-known Rossby number, which also happens to be the ratio of the magnitude of $\vec{u}_a$ to $\vec{u}$:

$$\mathrm{Ro} \sim \frac{U_a}{U}$$

Therefore, a small Ro means a small ageostrophic wind. This means, we cannot simply ignore $O(\mathrm{Ro})$ terms if we want to predict the time evolution of the flow!

Time Evolution of the Atmosphere

If we want to understand the time evolution of the atmosphere, we need to be able to solve for all of the unknowns. So far, we have 3 momentum equations (u, v, w), but 5 unknowns (u, v, w, ρ, p) (the unknowns appear in the momentum equations). While the ideal gas law allows us to relate ρ and p, it introduces another variable T (temperature). Therefore, we **need other equations** to be able to complete our mathematical description of the atmosphere. It turns out that these equations come from mass and energy.

Mass Continuity

Consider a fixed volume V with surface area A and outward unit normal $\hat{n}$. The mass contained in this volume is:

$$m = \int_V \rho\, dV$$

We can take the time derivative of the above:

$$\frac{\partial m}{\partial t} = \int_V \frac{\partial \rho}{\partial t}\, dV$$

Now, since the rate of change of mass in this volume can only occur from mass exchange with the environment, the mass flux out of the volume (F) is equivalent to:

$$F = \int_A (\rho \vec{u}) \cdot \hat{n}\, dA$$

Applying the divergence theorem to the flux,

$$F = \int_V \nabla \cdot (\rho \vec{u})\, dV,$$

Since V is fixed in space, the change in mass can only be equal to the mass flux out of the volume:

$$\frac{\partial m}{\partial t} = -F.$$

Next, we can combine our two equations for $\dfrac{\partial m}{\partial t}$:

$$\int_V \frac{\partial \rho}{\partial t}\, dV = -\int_V \nabla \cdot (\rho \vec{u})\, dV \qquad \Longrightarrow \qquad \int_V \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u})\right] dV = 0.$$

This must hold for any arbitrary V, so

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0} \qquad \text{or also written as} \qquad \boxed{\frac{D\rho}{Dt} + \rho(\nabla \cdot \vec{u}) = 0}$$

This equation is mass continuity – it uses the law that mass is neither created nor destroyed. For any fixed (fluid-following or arbitrary) volume, the rate of change of mass inside must be balanced entirely by the net flux of mass across its boundary. No mass appears out of nowhere and no mass vanishes, as mass can only be redistributed by motion.

Energy (Thermodynamic)

Conservation of thermodynamic energy tells us that the potential temperature is conserved following the flow, or:

$$\boxed{\frac{D\theta}{Dt} = 0} \qquad \text{where} \qquad \theta(T, p) = T\left(\frac{p_0}{p}\right)^{R_d/c_p} \tag{4}$$

which you should know from Thermodynamics, Atmospheric Dynamics, and Synoptic I.

Optional Derivation of Potential Temperature. The first law of thermodynamics reads

$$dI = dQ - dW + dC,$$

where dI is internal energy, dQ is heat input, dW is work done on or by the parcel, and dC is change due to chemical composition. For an ideal gas, the work for volume change is the pressure times change in volume, $dW = p\, d\alpha$ with $\alpha = 1/\rho$, and for a parcel of fixed composition, we have $dC = 0$. Thus, for an ideal gas,

$$dQ = dI + p\, d\alpha.$$

At constant volume, $dI = c_v\,dT$ where c_v is the specific heat at constant volume. Using $\alpha = R_d T/p$ for an ideal gas,

$$p\,d\alpha = p\,d\left(\frac{R_d T}{p}\right) = R_d\,dT - \frac{R_d T}{p}\,dp,$$

so that with $c_p - c_v = R_d$,

$$dQ = c_v\,dT + R_d\,dT - \frac{R_d T}{p}\,dp \qquad \Longrightarrow \qquad \boxed{dQ = c_p\,dT - \alpha\,dp}$$

which is defined in (T, p) coordinates. For an adiabatic process, $dQ = 0$:

$$c_p\,dT = \alpha\,dp \qquad \Longrightarrow \qquad \frac{dT}{T} = \frac{\alpha}{c_p}\frac{dp}{T} = \frac{R_d}{c_p}\frac{dp}{p}.$$

where we used the ideal gas law $p = \rho R_d T$. Integrating from p_0 to p,

$$\int_{p_0}^{p} \frac{dT}{T} = \frac{R_d}{c_p}\int_{p_0}^{p}\frac{dp}{p} \qquad \Longrightarrow \qquad \ln\left(\frac{T(p)}{T(p_0)}\right) = \frac{R_d}{c_p}\ln\left(\frac{p}{p_0}\right),$$

which when integrated yields the potential temperature defined in Eq. 4.

The Dry Primitive Equations

We now have a complete system consisting of 7 equations and 7 unknowns. There are three momentum equations, one equation for mass, one equation for thermodynamic energy, and one state equation (the ideal gas law). This is a solvable system! Unfortunately, there is no closed form, analytic solution to this system of equations, since they are nonlinear (note the presence of the material derivative). They must be solved numerically, which is exactly what numerical weather prediction (NWP) models do.

$$\frac{Du}{Dt} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{5}$$

$$\frac{Dv}{Dt} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y} \tag{6}$$

$$\frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g \tag{7}$$

$$\frac{D\theta}{Dt} = 0 \tag{8}$$

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{u}) = 0 \tag{9}$$

$$p = \rho R_d T \tag{10}$$

$$\theta = T\left(\frac{p_0}{p}\right)^{R_d/c_p} \tag{11}$$

This set is often called the dry primitive equations (or the compressible, non-hydrostatic equations of motion). Much of the rest of this course is about deriving simplified versions of this full system that are more tractable and that isolate the dynamics relevant to synoptic-scale motion, by exploiting the scale analysis that we've been doing.

The Anelastic Approximation

The first approximation we will make is called the anelastic approximation. It takes advantage of the fact that density variations in the atmosphere are, to leading order, small compared to the background density profile. Furthermore, since we are not interested in resolving acoustic (sound) waves, which are a fast, dynamically unimportant byproduct of the full compressible equations. To do this, let us first decompose the density and pressure into a reference profile that depends only on z, plus a small perturbation (this method is called "linearization"):

$$\left.\begin{aligned}\rho &= \bar{\rho}(z) + \rho'(x,y,z,t)\\ p &= \bar{p}(z) + p'(x,y,z,t)\end{aligned}\right\} \qquad \text{where} \quad \rho' \ll \bar{\rho}.$$

We will use this approximation to massage the primitive equations.

Horizontal Momentum Equations

Consider the horizontal pressure gradient force. Since $\rho' \ll \bar{\rho}$,

$$\frac{1}{\bar{\rho}+\rho'}\nabla(\bar{p}+p') \;\approx\; \frac{1}{\bar{\rho}}\nabla(\bar{p}+p').$$

Since $\bar{\rho}$ and $\bar{p}$ are not functions of t (and horizontal derivatives of $\bar{p}(z)$ vanish), the horizontal momentum equations become:

$$\frac{D\vec{u}}{Dt} + f\hat{k}\times\vec{u} = -\frac{1}{\bar{\rho}}\nabla p' \qquad \text{(horizontal only)}$$

which we can simplify a bit further by defining the scaled pressure perturbation $\phi \equiv \frac{p'}{\bar{\rho}}$:

$$\boxed{\frac{Du}{Dt} - fv = -\frac{\partial\phi}{\partial x}} \tag{12}$$

$$\boxed{\frac{Dv}{Dt} + fu = -\frac{\partial\phi}{\partial y}} \tag{13}$$

Continuity Equation

Let us start from the full continuity equation,

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{u}) = 0.$$

and substitute $\rho = \bar{\rho}(z) + \rho'(x,y,z,t)$. Since $\bar{\rho}$ is independent of t, $\partial\bar{\rho}/\partial t = 0$,

$$\frac{\partial\rho'}{\partial t} + \nabla\cdot\big[(\bar{\rho}+\rho')\vec{u}\big] = 0$$

Expanding the divergence,

$$\frac{\partial\rho'}{\partial t} + \frac{\partial}{\partial x}\big[(\bar{\rho}+\rho')u\big] + \frac{\partial}{\partial y}\big[(\bar{\rho}+\rho')v\big] + \frac{\partial}{\partial z}\big[(\bar{\rho}+\rho')w\big] = 0$$

and since density perturbations ρ' are small with respect to $\bar{\rho}$, we drop the ρ' advection terms:

$$\frac{\partial \rho'}{\partial t} + \bar{\rho}\big(\nabla_H \cdot \vec{u}\big) + \frac{\partial}{\partial z}(\bar{\rho}\, w) = 0.$$

Dropping the tendency term $\partial\rho'/\partial t$ removes "compressibility" and filters out sound waves, leaving the anelastic continuity equation:

$$\boxed{\bar{\rho}\big(\nabla_H \cdot \vec{u}\big) + \frac{\partial}{\partial z}(\bar{\rho}\, w) = 0} \qquad \text{or just} \qquad \boxed{\nabla \cdot (\bar{\rho}\, \vec{u}) = 0} \tag{14}$$

Before we proceed, let us scale this anelastic continuity equation, as it allows us to understand how w scales with the horizontal wind. Writing the total horizontal wind as $\vec{u} = \vec{u}_g + \vec{u}_a$, the horizontal divergence splits as:

$$\nabla_H \cdot \vec{u} = \nabla_H \cdot \vec{u}_g + \nabla_H \cdot \vec{u}_a.$$

Since $\nabla_H \cdot \vec{u}_g = 0$ (if f is not a function of y), only the *ageostrophic* part contributes to the horizontal divergence. Therefore, we have

$$\bar{\rho}\, \nabla_H \cdot \vec{u}_a + \frac{\partial}{\partial z}(\bar{\rho}\, w) = 0$$

which, recalling $\mathrm{Ro} \sim U_a/U$, allows us to write the scaling:

$$\frac{U_a}{L} \sim \frac{W}{H} \qquad \Longrightarrow \qquad \boxed{\frac{\mathrm{Ro}\, U}{L} \sim \frac{W}{H}}$$

Using $U \sim 10$ m/s, $L \sim 1000$ km (synoptic), $H \sim 10$ km (depth of troposphere), $f \sim 10^{-4}$ s^{-1} (mid-latitude), and $\mathrm{Ro} \sim 0.1$:

$$\frac{0.1 \cdot 10 \text{ m/s} \cdot 10^4 \text{ m}}{10^6 \text{ m}} = W \sim 0.01 \text{ m/s} \approx 1 \text{ cm/s}.$$

This tells us that the magnitude of vertical velocity associated with synoptic-scale motion is around 1 cm/s (we just naively assumed this before)!

Vertical Momentum Equation

We next apply the anelastic approximation to the vertical momentum equation. If we apply $p = \bar{p}(z) + p'(x, y, z, t)$ and $\rho = \bar{\rho}(z) + \rho'(x, y, z, t)$, then

$$\frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial(\bar{p} + p')}{\partial z} - g.$$

Multiplying through by ρ and substituting $\rho = \bar{\rho} + \rho'$ into the gravity term,

$$\rho\, \frac{Dw}{Dt} = -\frac{\partial \bar{p}}{\partial z} - \bar{\rho} g \; - \; \frac{\partial p'}{\partial z} - \rho' g.$$

Next, let us assume the reference profile follows hydrostatic balance (this is a very good assumption), or

$$\frac{\partial \bar{p}}{\partial z} = -\bar{\rho} g,$$

so that the reference-state terms cancel exactly:

$$-\frac{\partial \bar{p}}{\partial z} - \bar{\rho} g = 0.$$

This leaves

$$\rho \frac{Dw}{Dt} = -\frac{\partial p'}{\partial z} - \rho' g.$$

Finally, consistent with the anelastic approximation ($\rho' \ll \bar{\rho}$), we approximate $\rho \approx \bar{\rho}$ on the left-hand side:

$$\bar{\rho} \frac{Dw}{Dt} = -\frac{\partial p'}{\partial z} - \rho' g$$

$$\boxed{\frac{Dw}{Dt} = -\frac{1}{\bar{\rho}} \frac{\partial p'}{\partial z} - \frac{\rho'}{\bar{\rho}} g} \tag{15}$$

The vertical acceleration is therefore driven only by the *perturbation* pressure gradient and the *perturbation* (buoyancy) density. In other words, the reference hydrostatic state exerts no net force, as expected, since it is balanced by construction. Unfortunately, this equation is not entirely useful to us, since in Eq. 12, we scaled the pressure term by the reference density. Therefore, we need to rewrite this equation to be only a function of ϕ. To do this, we start from the chain rule,

$$\frac{\partial}{\partial z}\left(\frac{p'}{\bar{\rho}}\right) = \frac{1}{\bar{\rho}} \frac{\partial p'}{\partial z} - \frac{p'}{\bar{\rho}^2} \frac{\partial \bar{\rho}}{\partial z} \qquad \Longrightarrow \qquad -\frac{1}{\bar{\rho}} \frac{\partial p'}{\partial z} = -\frac{\partial}{\partial z}\left(\frac{p'}{\bar{\rho}}\right) - \frac{p'}{\bar{\rho}^2} \frac{\partial \bar{\rho}}{\partial z}.$$

Substituting into Eq. 15,

$$\frac{Dw}{Dt} = -\frac{\partial}{\partial z}\left(\frac{p'}{\bar{\rho}}\right) - \frac{p'}{\bar{\rho}^2} \frac{\partial \bar{\rho}}{\partial z} - \frac{\rho'}{\bar{\rho}} g \qquad \Longrightarrow \qquad \frac{Dw}{Dt} = -\frac{\partial \phi}{\partial z} - \frac{p'}{\bar{\rho}^2} \frac{\partial \bar{\rho}}{\partial z} - \frac{\rho'}{\bar{\rho}} g.$$

For synoptic scales Dw/Dt is small (can you do the scaling?), so

$$\boxed{-\frac{\partial \phi}{\partial z} - \frac{p'}{\bar{\rho}^2} \frac{\partial \bar{\rho}}{\partial z} - \frac{\rho'}{\bar{\rho}} g = 0}$$

This is not in the form we want yet, since we want to relate ϕ to the variable that comes out of the thermodynamic equation. In homework, you will show that this reduces to:

$$\boxed{\frac{\partial \phi}{\partial z} - b = 0} \tag{16}$$

where $b = g\theta'/\bar{\theta}$ is buoyancy. Eq. 16 is the vertical momentum equation with the anelastic and hydrostatic approximations. Notice how much simpler it is than the original vertical momentum equation!

Thermodynamic Equation

The final equation we need to modify is the thermodynamic equation. Let us start from conservation of potential temperature under the anelastic approximation, with $\theta(x, y, z, t) = \bar{\theta}(z) + \theta'(x, y, z, t)$:

$$\frac{D(\bar{\theta} + \theta')}{Dt} = 0$$

which can be rewritten as

$$\frac{\partial \theta'}{\partial t} + \vec{u} \cdot \nabla_H \theta' + w \frac{\partial \bar{\theta}}{\partial z} + w \frac{\partial \theta'}{\partial z} = 0.$$

where $\nabla_H \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right)$ is the horizontal gradient operator. We want to get this in terms of the buoyancy, $b = g\theta'/\bar{\theta}$. Let's multiply by g and divide by $\bar{\theta}$:

$$\frac{\partial}{\partial t}\left(\frac{g\theta'}{\bar{\theta}} \right) + \vec{u} \cdot \nabla_H \left(\frac{g\theta'}{\bar{\theta}} \right) + \underbrace{w \frac{g}{\bar{\theta}} \frac{\partial \bar{\theta}}{\partial z}}_{(1)} + \underbrace{w \frac{g}{\bar{\theta}} \frac{\partial \theta'}{\partial z}}_{(2)} = 0.$$

Since $\theta' \ll \bar{\theta}$, we will keep term (1) and discard term (2) (you will do this more rigorously in the homework). Defining the buoyancy $b = g\theta'/\bar{\theta}$ and the dry stratification

$$\frac{g}{\bar{\theta}} \frac{\partial \bar{\theta}}{\partial z} = N^2 \approx 10^{-4} \text{ s}^{-2},$$

we obtain the thermodynamic (buoyancy) equation under the anelastic approximation

$$\boxed{\frac{D_H b}{Dt} + wN^2 = 0} \tag{17}$$

where $\frac{D_H}{Dt} \equiv \frac{\partial}{\partial t} + \vec{u} \cdot \nabla_H$ is the material derivative following the *horizontal* flow only.

The Anelastic, Hydrostatic Primitive Equations

Equations 12, 13, 14, 16, 17, make the anelastic, hydrostatic primitive equations.

The Hydrostatic, Anelastic Equations

$$1) \quad \frac{Du}{Dt} - fv = -\frac{\partial \phi}{\partial x}$$

$$2) \quad \frac{Dv}{Dt} + fu = -\frac{\partial \phi}{\partial y}$$

$$3) \quad \frac{\partial \phi}{\partial z} = b$$

$$4) \quad \frac{D_H b}{Dt} + wN^2 = 0$$

$$5) \quad \nabla \cdot (\bar{\rho}\, \vec{u}) = 0$$

where $\phi = \frac{p'}{\bar{\rho}}$ is the (scaled) pressure and $b = g\frac{\theta'}{\bar{\theta}}$ is the buoyancy, with

$$\rho = \bar{\rho}(z) + \rho'(x, y, z, t), \qquad \theta = \bar{\theta}(z) + \theta'(x, y, z, t).$$

There are 5 equations and 5 unknowns (u, v, w, ϕ, b), thus making a **a complete solvable system**! The rest of this course will use these equations to understand synoptic-scale motion.

The Quasi-Geostrophic Approximation

Chapter 5

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Reference: Vallis, Chapter 5.4

Where We Left Off

In the previous classes, we assumed the following for synoptic-scale flow:

1) A small Rossby number, Ro $\ll 1$, so that the flow is approximated by geostrophic balance. This also means $\vec{u}_a \ll \vec{u}_g$. However, as we found, the ageostrophic wind is small but not necessarily negligible.

2) An advective time scale, $\tau = \dfrac{L}{U}$.

3) A density profile that can be decomposed into a reference profile plus a small perturbation (the anelastic approximation), $\rho(x, y, z, t) = \bar{\rho}(z) + \rho'(x, y, z, t)$.

4) Hydrostatic balance holds in the reference atmosphere

Under these assumptions, the dry, primitive equations reduced to the hydrostatic, anelastic equations:

The Hydrostatic, Anelastic Equations

$$\text{1,2)} \quad \frac{D\vec{u}}{Dt} + f_0\,\hat{k} \times \vec{u} = -\nabla_H \phi$$

$$\text{3)} \quad \frac{\partial \phi}{\partial z} = b$$

$$\text{4)} \quad \frac{D_H b}{Dt} + wN^2 = 0$$

$$\text{5)} \quad \bar{\rho}\,(\nabla_H \cdot \vec{u}) + \frac{\partial}{\partial z}\big(\bar{\rho}\,w\big) = 0$$

where $\phi = \dfrac{p'}{\bar{\rho}}$ is the (scaled) pressure and $b = g\dfrac{\theta'}{\bar{\theta}}$ is the buoyancy, with

$$\rho = \bar{\rho}(z) + \rho'(x, y, z, t), \qquad \theta = \bar{\theta}(z) + \theta'(x, y, z, t).$$

Note how there are 5 equations and 5 unknowns (u, v, w, ϕ, b) – a complete, solvable system! What we want to do now is to **analyze and solve these equations by making them appropriate for synoptic-scale storms**.

Non-dimensionalization

First, in order to make these appropriate for synoptic-scale storms, we will non-dimensionalize the equations. Non-dimensionalization is the process of making each term in an equation have no dimension, so that the relative size of each term is immediately apparent. We have already done something very similar before (scale analysis). Recall the scaling for the horizontal momentum:

$$\begin{array}{ccccccc} \dfrac{\partial u}{\partial t} & + & (\vec{u}\cdot\nabla)u & - fv & = & -\dfrac{\partial \phi}{\partial x} \\ \dfrac{U^2}{L} & & \dfrac{U^2}{L} & fU & & \dfrac{\phi}{L} \\ \dfrac{U}{fL} & & \dfrac{U}{fL} & 1 & & \dfrac{\phi}{fUL} \\ \mathrm{Ro} & & \mathrm{Ro} & 1 & & \dfrac{\phi}{fUL} \end{array}$$

where the second line divides through by fU. The first two terms scale as the Ro, the third term by 1, and the last term by ϕ/fUL. From this we concluded that if $\mathrm{Ro} \ll 1$, then geostrophic balance holds (i.e. the third term and RHS balance).

Choosing the Scales

To non-dimensionalize, we have to choose how to non-dimensionalize – that is, we must pick a characteristic scale for each variable and write each variable as a (scale) $\times$ (non-dimensional variable), the latter denoted with a hat. In other words:

$$\begin{aligned} (x, y) &\longrightarrow L\hat{x},\ L\hat{y} \\ (u, v) &\longrightarrow U\hat{u},\ U\hat{v} \\ t &\longrightarrow \frac{L}{U}\hat{t} \\ f &\longrightarrow f_0\hat{f} \\ \phi &\longrightarrow f_0 UL\,\hat{\phi} \end{aligned}$$

where all hatted terms are non-dimensional (and $O(1)$), and the *dimensional* scales are L, U, and f_0. We will next use these non-dimensional transformations to scale the hydrostatic, anelastic equations.

Horizontal Momentum Equations

Substituting the above scalings into the horizontal momentum equation gives

$$\frac{U^2}{L}\frac{\partial \hat{u}}{\partial \hat{t}} + \frac{U^2}{L}(\hat{\vec{u}}\cdot\hat{\nabla})\hat{u} - f_0 U\hat{f}\hat{v} = -\frac{f_0 UL}{L}\frac{\partial \hat{\phi}}{\partial \hat{x}}.$$

Dividing through by $f_0 U$,

$$\frac{U}{f_0 L}\frac{\partial \hat{u}}{\partial \hat{t}} + \frac{U}{f_0 L}(\hat{\vec{u}}\cdot\hat{\nabla})\hat{u} - \hat{f}\hat{v} = -\frac{\partial \hat{\phi}}{\partial \hat{x}},$$

and recognizing $\mathrm{Ro} = U/(f_0 L)$,

$$\boxed{\mathrm{Ro}\,\frac{\partial \hat{u}}{\partial \hat{t}} + \mathrm{Ro}\,(\hat{\vec{u}}\cdot\hat{\nabla})\hat{u} - \hat{f}\hat{v} = -\frac{\partial \hat{\phi}}{\partial \hat{x}}} \tag{1}$$

These are the non-dimensional momentum equations – notice how the Rossby number naturally comes out in this equation, and multiplies exactly the terms we diagnosed with the scale analysis were proportional to the Rossby number. The Rossby number is a **natural** non-dimensional number governing the momentum equations.

Hydrostatic Equation

Let us next look at the hydrostatic equation, $\dfrac{\partial \phi}{\partial z} = b$. Here we need scales for z and b, which we take to be

$$z \longrightarrow H\hat{z}, \qquad b \longrightarrow \frac{f_0 U L}{H}\hat{b},$$

so that

$$\frac{f_0 U L}{H}\frac{\partial \hat{\phi}}{\partial \hat{z}} = \frac{f_0 U L}{H}\hat{b} \qquad \Longrightarrow \qquad \boxed{\frac{\partial \hat{\phi}}{\partial \hat{z}} = \hat{b}}$$

Note, there is no other sensible choice of the non-dimensionalization of b here, as the left-hand-side is determined by our choice of non-dimensionalizing ϕ and z. The hydrostatic relation is unchanged by non-dimensionalization – no small parameter appears in front of either term. (Why does this make sense?)

Thermodynamic Equation

Starting from $\dfrac{D_H b}{Dt} + wN^2 = 0$ and substituting the scales above,

$$\frac{f_0 U^2}{L}\frac{L}{H}\frac{D_H \hat{b}}{Dt} + N^2 w = 0.$$

Note, we have not chosen the non-dimensionalization of w yet! How should we choose the scale for w? The answer comes from mass conservation.

Continuity Equation

The anelastic continuity equation is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{\bar{\rho}}\frac{\partial(\bar{\rho}w)}{\partial z} = 0.$$

Non-dimensionalizing the horizontal terms (but leaving w dimensional),

$$\frac{U}{L}\frac{\partial \hat{u}}{\partial \hat{x}} + \frac{U}{L}\frac{\partial \hat{v}}{\partial \hat{y}} + \frac{1}{H}\frac{1}{\bar{\rho}}\frac{\partial \bar{\rho}w}{\partial \hat{z}} = 0,$$

and dividing by U/L,

$$\frac{\partial \hat{u}}{\partial \hat{x}} + \frac{\partial \hat{v}}{\partial \hat{y}} + \frac{L}{UH}\frac{1}{\bar{\rho}}\frac{\partial \bar{\rho}w}{\partial \hat{z}} = 0.$$

Therefore, if we choose

$$w \sim \frac{UH}{L}\hat{w}$$

then the continuity equation becomes completely non-dimensional:

$$\boxed{\frac{\partial \hat{u}}{\partial \hat{x}} + \frac{\partial \hat{v}}{\partial \hat{y}} + \frac{1}{\bar{\rho}}\frac{\partial(\bar{\rho}\hat{w})}{\partial \hat{z}} = 0} \tag{2}$$

However, our choice of how to non-dimensionalize w may not be optimal, and we will see the ramifications of that decision later.

Returning to the Thermodynamic Equation

Now that we have a scale for w, we can return to the thermodynamic equation:

$$\frac{f_0 U^2}{L}\frac{L}{H}\frac{D_H \hat{b}}{Dt} + \frac{N^2 UH}{L}\hat{w} = 0.$$

Dividing through by $f_0^2 UL/H$,

$$\frac{U}{f_0 L}\frac{D_H \hat{b}}{Dt} + \frac{N^2 H^2}{f_0^2 L^2}\hat{w} = 0,$$

i.e.

$$\boxed{\mathrm{Ro}\,\frac{D_H \hat{b}}{Dt} + \left(\frac{L_D}{L}\right)^2 \hat{w} = 0} \tag{3}$$

where we have defined

$$L_D \equiv \frac{NH}{f_0}. \tag{4}$$

which is called the deformation radius (can you notice what is weird about Eq. (3)?). The deformation radius is the **length scale** at which Coriolis and buoyancy effects are equally important in constraining vertical motion. Using Earth-like parameters of $N \sim 10^{-2}$ s^{-1}, $H \sim 10^4$ m, and $f_0 \sim 10^{-4}$ s^{-1}:

$$L_D \sim \frac{10^{-2} \cdot 10^4}{10^{-4}} = 10^6 \text{ m} \sim 1000 \text{ km},$$

which is exactly **synoptic scales**! It turns out, this is no mere coincidence.

What the Deformation Radius Tells Us

Suppose the length scale of a disturbance is much smaller than the deformation radius ($L \ll L_D$), or small scales. Then Eq. 3 forces $\hat{w} \approx 0$: there is **no vertical motion**!

Suppose instead that the length scale of a disturbance is much greater than the deformation radius ($L \gg L_D$, or large scales). Then the second term drops out, leaving $\frac{D_H b}{Dt} = 0$. In other words, buoyancy is simply advected by the horizontal flow, i.e. it is a passive tracer. The temperature does not change following the flow!

Finally, suppose that the length scale of the disturbance is approximately the same as the deformation radius ($L \sim L_D$). Then, both terms are the same order, and the vertical velocity and buoyancy (or temperature) are tied together through this equation. In other words, there can exist a two-way

coupling between the dynamics (u, v, w) and the thermodynamics (b). This two-way feedback – where buoyancy anomalies drive vertical velocity, which affects the flow, which affects buoyancy, etc. – is *central* to synoptic-scale storms, and one we will spend a great deal discussing in this course. This is the interesting regime, and it is the one the mid-latitude atmosphere lives in.

The Beta Plane

Finally, we allow the Coriolis parameter to vary with latitude:

$$f = f_0 + \beta y, \qquad \text{where} \quad \beta \ll f_0.$$

Here β is an approximation to the change in the Coriolis force with latitude:

$$\beta = \frac{df}{dy} = \frac{d}{dy}\Big(2\Omega \sin(\varphi)\Big) = \frac{2\Omega \cos(\varphi)}{a} \sim 10^{-11}\ \frac{1}{\mathrm{s\,m}}.$$

We can use scaling to non-dimensionalize this too. Note that $\beta L \approx 10^{-5}\ \mathrm{s}^{-1}$ while $f_0 \approx 10^{-4}\ \mathrm{s}^{-1}$, so if $U \sim 10$ m/s and $L \sim 1000$ km $= 10^6$ m, then

$$\frac{U}{L^2} \approx 10^{-11}\ \frac{1}{\mathrm{s\,m}},$$

which is the same order as β. Therefore, assume

$$\beta \longrightarrow \frac{U}{L^2}\hat{\beta} = \frac{\mathrm{Ro}\, f_0}{L}\hat{\beta}, \qquad f \longrightarrow f_0 \hat{f}, \qquad y \longrightarrow L\hat{y}, \qquad \text{with} \quad \hat{f}_0 = 1.$$

Then $f = f_0 + \beta y$ becomes

$$f_0 \hat{f} = f_0 + \frac{\mathrm{Ro}\, f_0}{L}\hat{\beta}\, L\hat{y} \qquad \Longrightarrow \qquad \boxed{\hat{f} = \hat{f}_0 + \mathrm{Ro}\, \hat{\beta}\hat{y}}$$

In other words, **variations in f only matter if** Ro **is not small**.

The Non-dimensional Equations

Collecting everything (and now dropping the hats for convenience), we have the final non-dimensional equations:

The Non-dimensional Equations

1) $\mathrm{Ro}\,\dfrac{Du}{Dt} - fv = -\dfrac{\partial \phi}{\partial x}$

2) $\mathrm{Ro}\,\dfrac{Dv}{Dt} + fu = -\dfrac{\partial \phi}{\partial y}$

3) $\dfrac{\partial \phi}{\partial z} = b$

4) $\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{1}{\bar{\rho}}\dfrac{\partial(\bar{\rho} w)}{\partial z} = 0$

5) $\mathrm{Ro}\,\dfrac{Db}{Dt} + \left(\dfrac{L_D}{L}\right)^2 w = 0$

6) $f = f_0 + \mathrm{Ro}\,\beta y$

Asymptotic Expansions

Next, we will perform asymptotic expansions on the above equations, in order to understand which terms balance with which. Since Ro is a small parameter, we can expand each variable as an asymptotic series in Ro:

$$u = u_0 + \mathrm{Ro}\, u_1 + O(\mathrm{Ro}^2) + \dots$$
$$v = v_0 + \mathrm{Ro}\, v_1 + \dots$$
$$\phi = \phi_0 + \mathrm{Ro}\, \phi_1 + \dots$$

Because Ro is small, the $O(\mathrm{Ro}^2)$ terms will be very small, and we may safely discard them.

Expanding the Material Derivative

Before substituting into the momentum equations, we must also expand the material derivative operator itself, since it depends on the velocity field. Writing

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (\vec{u}\cdot\nabla) = \frac{\partial}{\partial t} + (u_0 + \mathrm{Ro}\, u_1)\frac{\partial}{\partial x} + (v_0 + \mathrm{Ro}\, v_1)\frac{\partial}{\partial y} + \dots,$$

we can group terms by order in Ro:

$$\frac{D}{Dt} = \underbrace{\left[\frac{\partial}{\partial t} + (\vec{u}_0\cdot\nabla)\right]}_{D_0/Dt} + \mathrm{Ro}\,\underbrace{[(\vec{u}_1\cdot\nabla)]}_{D_1/Dt} + O(\mathrm{Ro}^2),$$

so that

$$\frac{D}{Dt} = \frac{D_0}{Dt} + \mathrm{Ro}\,\frac{D_1}{Dt} + O(\mathrm{Ro}^2), \qquad \frac{D_0}{Dt} \equiv \frac{\partial}{\partial t} + (\vec{u}_0\cdot\nabla), \qquad \frac{D_1}{Dt} \equiv (\vec{u}_1\cdot\nabla).$$

Applying this to Du/Dt, and also expanding $u = u_0 + \mathrm{Ro}\, u_1 + \dots$,

$$\frac{Du}{Dt} = \left(\frac{D_0}{Dt} + \mathrm{Ro}\frac{D_1}{Dt}\right)(u_0 + \mathrm{Ro}\, u_1) = \frac{D_0 u_0}{Dt} + \mathrm{Ro}\left(\frac{D_0 u_1}{Dt} + \frac{D_1 u_0}{Dt}\right) + O(\mathrm{Ro}^2)$$

Horizontal Momentum Equations

Substituting the expansions (and $f = f_0 + \mathrm{Ro}\,\beta y$) into the non-dimensional u-momentum equation:

$$\mathrm{Ro}\,\frac{D_0 u_0}{Dt} + O(\mathrm{Ro}^2) - f_0 v_0 - \mathrm{Ro}\,\beta y\, v_0 - \mathrm{Ro}\, f_0 v_1 - \mathrm{Ro}^2\,\beta y\, v_1 = -\frac{\partial \phi_0}{\partial x} - \mathrm{Ro}\frac{\partial \phi_1}{\partial x}.$$

Keeping only $O(1)$ terms:

$$\left.\begin{aligned} f_0 v_0 &= \frac{\partial \phi_0}{\partial x} \\ f_0 u_0 &= -\frac{\partial \phi_0}{\partial y} \end{aligned}\right\} \quad \textbf{geostrophic balance!}$$

So the leading-order flow is exactly the geostrophic wind, as we expected.

Keeping only the $O(\mathrm{Ro})$ terms:

$$\left.\begin{aligned} \mathrm{Ro}\,\frac{D_0 u_0}{Dt} - \mathrm{Ro}\,f_0 v_1 - \mathrm{Ro}\,\beta y\, v_0 &= -\mathrm{Ro}\frac{\partial \phi_1}{\partial x} \\ \mathrm{Ro}\,\frac{D_0 v_0}{Dt} + \mathrm{Ro}\,f_0 u_1 + \mathrm{Ro}\,\beta y\, u_0 &= -\mathrm{Ro}\frac{\partial \phi_1}{\partial y} \end{aligned}\right\} O(\mathrm{Ro})$$

where

$$\frac{D_0}{Dt} \equiv \frac{D_g}{Dt} \equiv \frac{\partial}{\partial t} + (\vec{u}_g \cdot \nabla) \qquad \Longrightarrow \qquad \text{only } \vec{u}_g \text{ advects!}$$

Notice that **the evolution of the flow depends on the $O(\mathrm{Ro})$ terms** (or the ageostrophic wind) – exactly the point we made when we first introduced $\vec{u}_a$. Cleaning up, and recognizing that the zero-th order zonal and meridional winds are actually the **geostrophic** winds (u_g, v_g), and the first order zonal and meridional winds are the ageostrophic winds (u_a, v_a), we have:

$$\boxed{\frac{D_g u_g}{Dt} - f_0 v_a - \beta y\, v_g = -\frac{\partial \phi_1}{\partial x}} \tag{5}$$

$$\boxed{\frac{D_g v_g}{Dt} + f_0 u_a + \beta y\, u_g = -\frac{\partial \phi_1}{\partial y}} \tag{6}$$

These are the **quasi-geostrophic** momentum equations. These equations are called *quasi*-geostrophic because the flow is *almost* but not exactly in geostrophic balance. Notice how the velocity that advects momentum is taken to be just u_g – **only** the geostrophic wind advects momentum around. However, the evolution of that very geostrophic flow is controlled by the small ageostrophic (unbalanced) part. So "quasi-geostrophic" captures the idea that the dynamics are dominated by geostrophic balance (hence "geostrophic"), but retains just enough of the next-order *ageostrophic* physics to allow the flow to actually change in time.

Continuity Equation

Substituting the expansions into the continuity equation:

$$\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y} + \mathrm{Ro}\frac{\partial u_1}{\partial x} + \mathrm{Ro}\frac{\partial v_1}{\partial y} + \frac{1}{\bar{\rho}}\frac{\partial(\bar{\rho} w_0)}{\partial z} + \mathrm{Ro}\,\frac{1}{\bar{\rho}}\frac{\partial(\bar{\rho} w_1)}{\partial z} = 0.$$

Keeping only $O(1)$ terms:

$$\underbrace{\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y}}_{=0} + \frac{1}{\bar{\rho}}\frac{\partial(\bar{\rho} w_0)}{\partial z} = 0 \qquad \Longrightarrow \qquad \frac{\partial(\bar{\rho} w_0)}{\partial z} = 0$$

since the geostrophic wind is horizontally non-divergent. Note that if we integrate the above, we can conclude that **if** $w_0 = 0$ **somewhere, then** $w_0 = 0$ **everywhere**, since the derivative of w_0 must be zero. Since w_0 must be zero at the surface, this leads us to conclude that $w_0 = 0$. The "geostrophic" vertical velocity wind is zero! We could have avoided this solution if we non-dimensionalized w as:

$$w \sim Ro\frac{UH}{L}\hat{w}$$

which is exactly the scaling for w that we found in the anelastic, incompressible continuity equation.

Keeping only $O(\mathrm{Ro})$ terms:

$$\underbrace{\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y}}_{\text{ageostrophic wind}} + \underbrace{\frac{1}{\bar{\rho}} \frac{\partial (\bar{\rho} w_1)}{\partial z}}_{\text{vertical velocity}} = 0$$

This equation says that the horizontal divergence of the ageostrophic wind is exactly balanced by the vertical stretching of w_1. Since we've already shown $w_0 = 0$, the vertical velocity w_1 is the actual, physically meaningful vertical velocity in this system. This makes physical sense: vertical motion in the atmosphere is **not** produced by the geostrophic wind (which is horizontally non-divergent by construction), but instead arises entirely from the small departures from geostrophic balance (i.e. the ageostrophic wind). Wherever the ageostrophic wind converges horizontally, there must be upwards motion, whereas wherever it diverges, there must be downwards motion. Cleaning up, we have the **quasi-geostrophic continuity equation**.

$$\boxed{\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y} + \frac{1}{\bar{\rho}} \frac{\partial (\bar{\rho} w)}{\partial z} = 0} \tag{7}$$

Thermodynamic Equation

Finally, starting from the non-dimensional thermodynamic equation, we substitute the expansions $b = b_0 + \mathrm{Ro}\, b_1 + \ldots$ and $w = w_0 + \mathrm{Ro}\, w_1 + \ldots$. Note that unlike the momentum equation, the material derivative here carries the horizontal subscript H (for horizontal advection only) but is otherwise expanded the same way:

$$\frac{D_H}{Dt} = \frac{D_{H,g}}{Dt} + \mathrm{Ro} \frac{D_{H,a}}{Dt} + O(\mathrm{Ro}^2)$$

so that

$$\frac{D_H b}{Dt} = \frac{D_{H,g} b_0}{Dt} + \mathrm{Ro} \left(\frac{D_{H,g} b_1}{Dt} + \frac{D_{H,a} b_0}{Dt} \right) + O(\mathrm{Ro}^2).$$

Substituting this, together with $w = w_0 + \mathrm{Ro}\, w_1 + \ldots$ (even though we already know $w_0 = 0$), into the full thermodynamic equation gives

$$\mathrm{Ro} \frac{D_{H,g} b_0}{Dt} + O(\mathrm{Ro}^2) + \left(\frac{L_D}{L} \right)^2 w_0 + \mathrm{Ro} \left(\frac{L_D}{L} \right)^2 w_1 = 0,$$

Keeping only $O(1)$ terms:

$$\frac{L_D^2}{L^2} w_0 = 0 \qquad \Longrightarrow \qquad w_0 = 0,$$

i.e. the lowest order w is zero (which is consistent with the corresponding result of the scaling of the continuity equation). In other words, there is **no "geostrophic" vertical velocity**.

Keeping only $O(\mathrm{Ro})$ terms:

$$\boxed{\frac{D_{g,H}}{Dt} b_0 + w \frac{L_D^2}{L^2} = 0} \tag{8}$$

where $D_{g,H}/Dt \equiv \partial_t + (\vec{u}_0 \cdot \nabla_H)$, or only the geostrophic wind advects buoyancy at this order. This is the **quasi-geostrophic thermodynamic equation**, which tells us that **the time evolution of temperature depends on the vertical velocity**! Just as with the momentum equations, it is the small, $O(\mathrm{Ro})$ part of the flow (w), and not the leading-order geostrophic part, that determines how the (temperature of the) atmosphere evolves in time.

Summary: The Quasi-Geostrophic Equations

The $O(\text{Ro})$ equations, together with the $O(1)$ geostrophic and hydrostatic balances that define $\vec{u}_g$ and b_0, form the quasi-geostrophic system:

The Quasi-Geostrophic Equations

$$1) \quad \frac{D_g u_g}{Dt} - f_0 v_a - \beta y\, v_g = -\frac{\partial \phi_1}{\partial x}$$

$$2) \quad \frac{D_g v_g}{Dt} + f_0 u_a + \beta y\, u_g = -\frac{\partial \phi_1}{\partial y}$$

$$3) \quad \frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y} + \frac{1}{\bar{\rho}} \frac{\partial (\bar{\rho} w)}{\partial z} = 0$$

$$4) \quad \frac{D_g b_0}{Dt} + w\, \frac{L_D^2}{L^2} = 0$$

where only the geostrophic wind advects, $\dfrac{D_g}{Dt} = \dfrac{\partial}{\partial t} + \vec{u}_g \cdot \nabla_H$, and the geostrophic wind and buoyancy are defined by the $O(1)$ balances

$$f_0 v_g = \frac{\partial \phi_0}{\partial x}, \qquad f_0 u_g = -\frac{\partial \phi_0}{\partial y}, \qquad \frac{\partial \phi_0}{\partial z} = b_0.$$

Vorticity and Quasi-Geostrophic Vorticity

Chapter 6

Last Modified: November 15, 2026

Where We Left Off

In the last class, we did an asymptotic expansion of the horizontal momentum equations, and separated terms by the order of their Rossby number. The $O(1)$ terms represented **geostrophic balance**:

$$\boxed{f_0 v_0 = \frac{\partial \phi_0}{\partial x}} \tag{1}$$

$$\boxed{f_0 u_0 = -\frac{\partial \phi_0}{\partial y}} \tag{2}$$

such that we concluded that $u_0 = u_g$ and $v_0 = v_g$. The $O(\text{Ro})$ led to the quasi-geostrophic momentum equations:

$$\boxed{\frac{D_g u_g}{Dt} - f_0 v_a - \beta y\, v_g = -\frac{\partial \phi_1}{\partial x}} \tag{3}$$

$$\boxed{\frac{D_g v_g}{Dt} + f_0 u_a + \beta y\, u_g = -\frac{\partial \phi_1}{\partial y}} \tag{4}$$

Our goal now is to combine both of these equations into a single equation for the **vorticity**, a measure of the local rotation (spin) of the atmosphere.

The Geostrophic Vorticity

Let us take the cross-derivative of the $O(1)$ (geostrophic) momentum equations. That is, we take $\partial/\partial x$ of the first and $\partial/\partial y$ of the second:

$$\frac{\partial}{\partial x}\left(f_0 v_0 = \frac{\partial \phi_0}{\partial x}\right) \qquad \Longrightarrow \qquad f_0 \frac{\partial v_0}{\partial x} = \frac{\partial^2 \phi_0}{\partial x^2}$$

$$\frac{\partial}{\partial y}\left(f_0 u_0 = -\frac{\partial \phi_0}{\partial y}\right) \qquad \Longrightarrow \qquad f_0 \frac{\partial u_0}{\partial y} = -\frac{\partial^2 \phi_0}{\partial y^2}$$

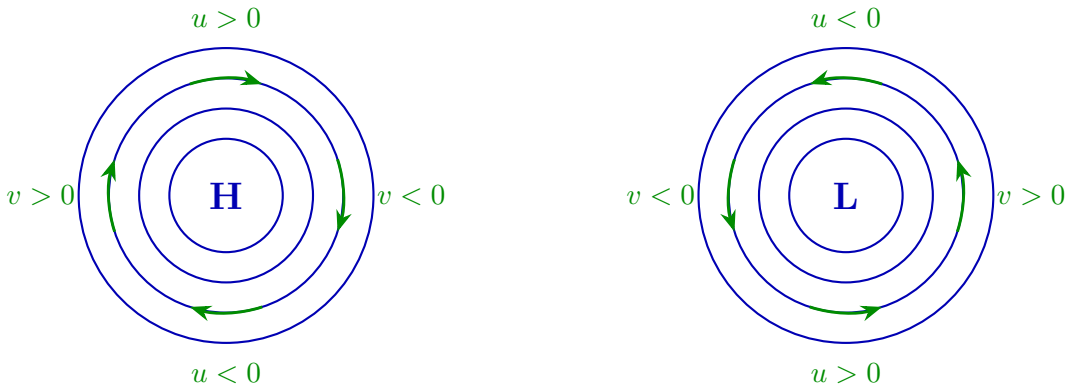
Subtracting the second from the first, we can derive an equation for the curl of the atmosphere, or the vorticity:

$$\zeta_0 \equiv \frac{\partial v_0}{\partial x} - \frac{\partial u_0}{\partial y} = \frac{1}{f_0}\left(\frac{\partial^2 \phi_0}{\partial x^2} + \frac{\partial^2 \phi_0}{\partial y^2}\right)$$

or generalizing to the geostrophic wind and vorticity:

$$\boxed{\zeta_g = \hat{k} \cdot (\nabla_H \times \vec{u}_g) = \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} = \frac{1}{f_0}\nabla_H^2 \phi} \tag{5}$$

Vorticity is a measure of how much the atmosphere is spinning. A **low** pressure system has **positive** vorticity, and a **high** pressure system has **negative** vorticity. We can understand why using the schematics below.



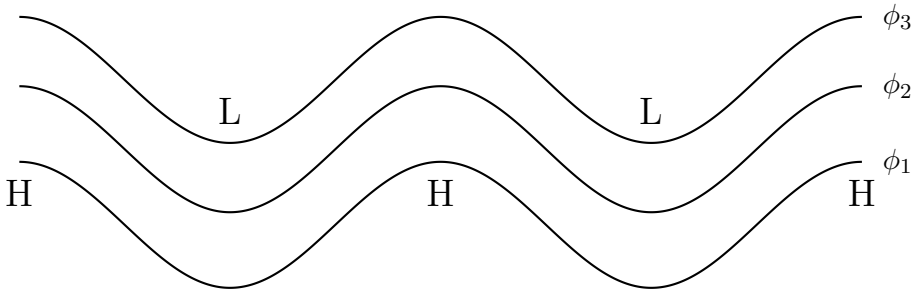
$$\frac{\partial u}{\partial y} > 0, \qquad \frac{\partial v}{\partial x} < 0$$

$$\zeta_g = (-) - (+) < 0$$

$$\frac{\partial u}{\partial y} < 0, \qquad \frac{\partial v}{\partial x} > 0$$

$$\zeta_g = (+) - (-) > 0$$

As we see in Eq. (5), the geostrophic vorticity is also related to the horizontal Laplacian of pressure. Recall that the Laplacian measures the curvature (or concavity) of a profile. Consider the following wavy geopotential pattern, where $\phi_1 > \phi_2 > \phi_3$:



At the crests of the wave-like patterns, the profile is concave down (a local maximum), such that $\nabla^2\phi < 0$. Thus, these areas of local high pressures (labeled **H**) are associated with *negative* vorticity. In contrast, at the troughs of the wave-like patterns, the profile is concave up (a local minimum), such that $\nabla^2\phi > 0$. Thus, these areas of local low pressures (labeled **L**) are associated with *positive* vorticity. This areas of local high pressures are called **ridges**, while the areas of local low pressures are called **troughs**. The former are associated with negative vorticity and clock-wise motion in the Northern Hemisphere, while the latter are associated with positive vorticity and counter clock-wise motion in the Northern Hemisphere.

Exercise: Can you draw the geostrophic wind in the above setup, and diagnose the vorticity from the direction of local rotation?

However, it is important to note that Eq. (5) is **purely** a balance equation. It does not tell you anything about how the flow will evolve in time. It can only tell you, instantaneously, the relationship between the geostrophic vorticity and the pressure (or geopotential). If we want to actually predict the **time evolution** of the flow, we need to turn to the $O(\mathrm{Ro})$ terms.

The Quasi-Geostrophic Vorticity Equation

Let us return to the asymptotic expansion of the horizontal momentum equations. These are $O(\text{Ro})$, and after dividing through by Ro:

$$\left.\begin{aligned} \frac{D_g u_g}{Dt} - f_0 v_a - \beta y\, v_g &= -\frac{\partial \phi_1}{\partial x} \\ \frac{D_g v_g}{Dt} + f_0 u_a + \beta y\, u_g &= -\frac{\partial \phi_1}{\partial y} \end{aligned}\right\} \quad \text{cross-differentiate}$$

Taking $\partial/\partial y$ of the first equation and $\partial/\partial x$ of the second:

$$1) \quad \frac{\partial}{\partial y}\left(\frac{D_g u_g}{Dt}\right) - f_0 \frac{\partial v_a}{\partial y} - \beta v_g - \beta y \frac{\partial v_g}{\partial y} = -\frac{\partial^2 \phi_1}{\partial x \partial y}$$

$$2) \quad \frac{\partial}{\partial x}\left(\frac{D_g v_g}{Dt}\right) + f_0 \frac{\partial u_a}{\partial x} + \beta y \frac{\partial u_g}{\partial x} = -\frac{\partial^2 \phi_1}{\partial x \partial y}$$

Subtracting, 2) − 1), the pressure terms cancel exactly and we are left with

$$\frac{D_g}{Dt}\left(\frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y}\right) + f_0 \left(\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y}\right) + \beta v_g + \beta y \underbrace{\left(\frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y}\right)}_{=0} = 0$$

where the last term vanishes because the geostrophic wind is horizontally non-divergent. Recognizing the first term as the material derivative of the geostrophic vorticity, we have:

$$\boxed{\frac{\partial \zeta_g}{\partial t} + (\vec{u}_g \cdot \nabla)\, \zeta_g + \beta v_g = -f_0\, (\nabla_H \cdot \vec{u}_a)} \tag{6}$$

This is the **quasi-geostrophic vorticity equation**, and our first, simplified equation that tells us how the atmosphere (more specifically, the geostrophic vorticity) evolves in time. However, the evolution of the geostrophic vorticity depends on $\vec{u}_a$, or the ageostrophic velocity! More specifically, it depends on the horizontal divergence of the ageostrophic velocity. Recall that the ageostrophic velocity is an $O(\text{Ro})$ term. Thus, this equation is called "quasi-geostrophic", since it includes a correction to geostrophy, namely through the divergence of the ageostrophic velocity. This is an again an example of how the small terms matter for the evolution of the total flow of the atmosphere.

It is important to note that this is not quite a complete system. While the left-hand-side is only a function of the geostrophic velocities and vorticity (the latter which itself is a function of the former), the right-hand-side is completely dependent on the ageostrophic velocity. Thus, if we wanted to complete this system, we would need a way to describe the ageostrophic velocities. More on this later!

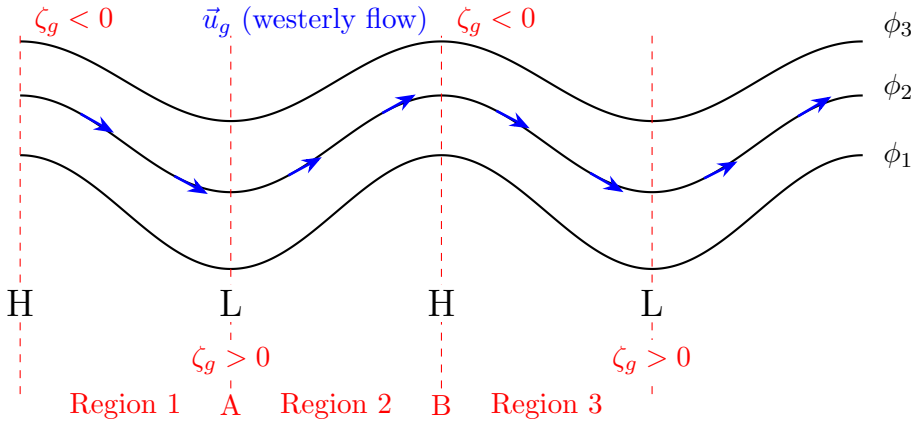
Let's rewrite the vorticity equation by combining the relative vorticity advection ($(\vec{u}_g \cdot \nabla)\zeta_g$) and planetary vorticity (βv_g) into a single advection term:

$$\frac{\partial \zeta_g}{\partial t} = \underbrace{-\,(\vec{u}_g \cdot \nabla)\,(\zeta_g + f)}_{\text{vorticity advection}} \underbrace{-\,f_0\,(\nabla_H \cdot \vec{u}_a)}_{\text{vorticity stretching}}, \qquad \text{where} \qquad f = f_0 + \beta y$$

where we have used the fact that planetary vorticity increases with latitude, so that $(\vec{u}_g \cdot \nabla) f = \beta v_g$.

Understanding the Vorticity Advection Term

Let's try to understand the vorticity advection term first. Again suppose an idealized wavy state of the atmosphere in the 500-hPa geopotential, where $\phi_1 > \phi_2 > \phi_3$:



We know that there the geostrophic wind is westerly (blue arrows), following the lines of constant geopotential. The relative vorticity is the Laplacian of the geopotential, so it is most **cyclonic** ($\zeta_g > 0$) at the trough axes (L) and most **anticyclonic** ($\zeta_g < 0$) at the ridge axes (H). This means that, moving from west to east, there is a gradient in vorticity, though the sign depends on where one is within the wave pattern.

Next, let us split the wave into two regions: **Region 1**, which lies between a ridge and the downstream trough, and **Region 2**, which lies between a trough and the downstream ridge.

Relative Vorticity Advection

Relative vorticity advection is quite literally, the advection of relative vorticity.

$$\text{Region 1:} \quad (\vec{u}_g \cdot \nabla)\,\zeta_g > 0 \quad \Longrightarrow \quad -(\vec{u}_g \cdot \nabla)\,\zeta_g < 0$$

$$\text{Region 2:} \quad (\vec{u}_g \cdot \nabla)\,\zeta_g < 0 \quad \Longrightarrow \quad -(\vec{u}_g \cdot \nabla)\,\zeta_g > 0$$

In **Region 1**, the wind is blowing across a region that increases from negative vorticity (ridge) to positive vorticity (trough). This means that the geostrophic wind is advecting negative vorticity, such that the relative vorticity advection term is negative, or $(\vec{u}_g \cdot \nabla)\,\zeta_g > 0$. The net effect of this is that over time, in region 1, the vorticity will tend to decrease. In other words, there will be an increase in the anticyclonic vorticity and a high pressure will build. This is called **anticyclonic vorticity advection (AVA)**.

In **Region 2**, the relative vorticity advection term is positive, because the wind is blowing from a positive vorticity region to a negative vorticity region. Therefore, in region 2, the vorticity will tend to increase. In other words, there will be an increase in the cyclonic vorticity and a low pressure will deepen. This is called **cyclonic vorticity advection (CVA)**.

Exercise: Can you deduce what the relative vorticity advection term is at points A (trough axis) and B (ridge axis)? What about in region 3?

Planetary Vorticity Advection

On the other hand, planetary vorticity advection is the advection of the Earth's background vorticity gradient; the Coriolis parameter increases as one moves from equator to pole, such that a southerly wind advects low planetary vorticity, while a northerly wind advects high planetary vorticity.

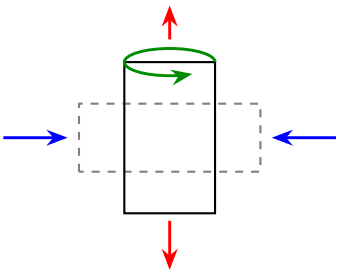
$$\text{Region 1:} \quad v_g < 0 \ \text{(southward flow)} \quad \Longrightarrow \quad -\beta v_g > 0$$

$$\text{Region 2:} \quad v_g > 0 \ \text{(northward flow)} \quad \Longrightarrow \quad -\beta v_g < 0$$

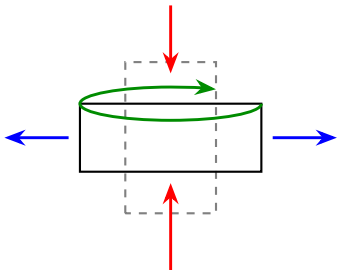
In **Region 1**, where the planetary advection term is positive, the vorticity will tend to increase, since southward flow advects **higher** planetary vorticity. In contrast, in **Region 2**, where lower planetary vorticity is advected from the south, the vorticity will tend to decrease. Notice how planetary advection has the opposite-signed effect of relative vorticity advection on vorticity tendency.

Vorticity Stretching

The last term in the QG vorticity equation, $f_0 \left(\nabla_H \cdot \vec{u}_a\right)$, is fundamentally different from the two advection terms. Vorticity stretching is the primary way that vorticity is amplified or weakened locally. Note also that this is the only term in the equation that involves the ageostrophic wind. We can use a figure-skater analogy to understand vorticity stretching. A rotating column of air that converges horizontally (owing to the ageostrophic wind), must, by mass continuity, stretch vertically. As it narrows, to conserve angular momentum it must rotate faster. In contrast, a rotating column of air the diverges horizontally, must, by the same logic, squeeze (or compress) vertically. As this column gets squeezed, it must rotate slower to conserve angular momentum.



$\nabla_H \cdot \vec{u}_a < 0$ (convergence)
$-f_0 \left(\nabla_H \cdot \vec{u}_a\right) > 0$
column stretches, ζ_g <u>increases</u>



$\nabla_H \cdot \vec{u}_a > 0$ (divergence)
$-f_0 \left(\nabla_H \cdot \vec{u}_a\right) < 0$
column squashes, ζ_g <u>decreases</u>

This term is one reason why surface cyclones deepen. Divergence aloft must be balanced by convergence at low levels, which in turn is associated with vorticity stretching near the surface. We will make this argument more quantitative in a few weeks. Notice, however, that we have not found yet how to compute $\vec{u}_a$ from the geostrophic flow alone. The system remains unclosed, and closing it will require bringing in the other the quasi-geostrophic thermodynamic equation.

<u>The Quasi-Geostrophic Vorticity Equation</u>

$$\frac{\partial \zeta_g}{\partial t} = \underbrace{-\left(\vec{u}_g \cdot \nabla\right)\zeta_g}_{\text{relative vorticity advection}} \qquad \underbrace{-\beta v_g}_{\text{planetary vorticity advection}} \qquad \underbrace{-f_0 \left(\nabla_H \cdot \vec{u}_a\right)}_{\text{vorticity stretching}}$$

where $\zeta_g = \dfrac{1}{f_0}\nabla_H^2 \phi$ is the geostrophic (relative) vorticity, and $f = f_0 + \beta y$.

Pressure Coordinates and the Quasi-Geostrophic Geopotential Tendency Equation

Chapter 7

Last Modified: November 15, 2026

Where We Left Off

In the last class, we showed that the geostrophic vorticity is related to the Laplacian of the geopotential,

$$\zeta_g = \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} = \frac{1}{f_0}\nabla_H^2 \phi,$$

and derived the quasi-geostrophic vorticity equation:

$$\boxed{\frac{\partial \zeta_g}{\partial t} = -\left(\vec{u}_g \cdot \nabla\right)\zeta_g - \beta v_g - f_0\left(\nabla_H \cdot \vec{u}_a\right)} \tag{1}$$

This is not really a closed equation, because it depends on $\vec{u}_a$ (the $O(\text{Ro})$ velocity $\vec{u}_1$), and we do not have an equation for that! Our goal in this lecture is to come up with a **closed-form** equation. How do we get $\vec{u}_a$? Let's return to our asymptotic expansion, this time of the continuity equation.

The Continuity Equation and the Ageostrophic Wind

Recall the (nondimensional) anelastic continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{\bar{\rho}}\frac{\partial(\bar{\rho}w)}{\partial z} = 0$$

We expanded each variable in a series in the Rossby number, e.g. $u = u_0 + \text{Ro}\, u_1 + O(\text{Ro}^2)$. Substituting,

$$\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y} + \frac{1}{\bar{\rho}}\frac{\partial(\bar{\rho}w_0)}{\partial z} + \text{Ro}\left(\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} + \frac{1}{\bar{\rho}}\frac{\partial(\bar{\rho}w_1)}{\partial z}\right) + O(\text{Ro}^2) = 0$$

Grouping the $O(\text{Ro})$ terms, and noting that $u_0 = u_g$, $v_0 = v_g$, and $w_1 = w$:

$$\underbrace{\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y}}_{\nabla_H \cdot \vec{u}_a} + \underbrace{\frac{1}{\bar{\rho}}\frac{\partial(\bar{\rho}w)}{\partial z}}_{\text{the } \textit{actual} \text{ vertical velocity}} = 0$$

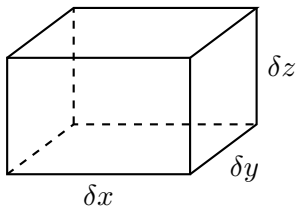
The divergence of the ageostrophic wind is therefore related to the vertical motion. This is exactly the vorticity stretching term in the QG vorticity equation. We can therefore substitute this relationship into the QG vorticity equation (Eq. 1):

$$\frac{\partial \zeta_g}{\partial t} = -\left(\vec{u}_g \cdot \nabla\right)\zeta_g - \beta v_g + f_0\left(\frac{1}{\bar{\rho}}\frac{\partial(\bar{\rho}w)}{\partial z}\right)$$

We have eliminated the ageostrophic velocity term, but now we need an equation for w, which itself appears in the $\partial(\bar{\rho}w)/\partial z$ term. We can further simplify the latter term by switching to **pressure coordinates** (as we did for the hydrostatic equation).

Pressure Coordinates

Pressure is a popular choice for the vertical coordinate in atmospheric dynamics. So far, we have used height z in the anelastic equations, though in the preceding lectures we already wrote hydrostatic balance in pressure coordinates. Consider a small control volume $\delta V = \delta x\, \delta y\, \delta z$:



Using hydrostatic balance,

$$\frac{\partial p}{\partial z} = -\rho g \qquad \Longrightarrow \qquad \delta z = -\frac{\delta p}{\rho g},$$

the mass of the control volume is

$$\delta M = \rho\, \delta V = \rho\, \delta x\, \delta y\, \delta z = -\frac{\delta x\, \delta y\, \delta p}{g}$$

Conservation of mass for this (material) control volume requires

$$\frac{1}{\delta M}\frac{D(\delta M)}{Dt} = \frac{g}{\delta x\, \delta y\, \delta p}\frac{D}{Dt}\left(\frac{\delta x\, \delta y\, \delta p}{g}\right) = 0 \qquad \Longrightarrow \qquad \frac{1}{\delta x\, \delta y\, \delta p}\frac{D}{Dt}\left(\delta x\, \delta y\, \delta p\right) = 0$$

Expanding with the product rule,

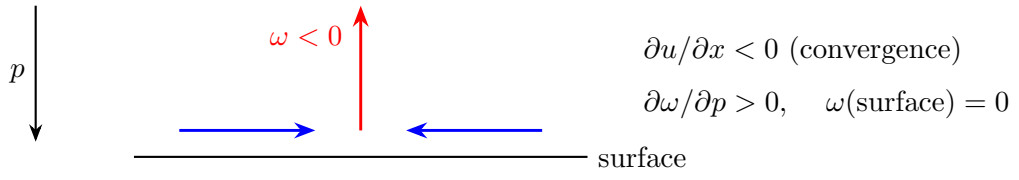
$$\frac{1}{\delta x}\frac{D(\delta x)}{Dt} + \frac{1}{\delta y}\frac{D(\delta y)}{Dt} + \frac{1}{\delta p}\frac{D(\delta p)}{Dt} = 0 \qquad \Longrightarrow \qquad \frac{\delta u}{\delta x} + \frac{\delta v}{\delta y} + \frac{\delta \omega}{\delta p} = 0$$

where $\omega \equiv Dp/Dt$ is the **pressure vertical velocity**. Taking the limit of an infinitesimal volume:

$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} = 0} \tag{2}$$

Notice that the continuity equation is much simpler in pressure coordinates. There is no density term, and the continuity equation in pressure coordinates looks like the incompressible continuity equation. Since pressure decreases with height, **upward** motion corresponds to $\omega < 0$ and **downward** motion corresponds to $\omega > 0$.

Interpretation:



Suppose that near the surface, where $\omega = 0$, there is horizontal convergence. This requires that $\partial\omega/\partial p > 0$, or $\omega < 0$ above the surface, i.e., rising motion.

The QG Vorticity Equation in Pressure Coordinates

We now rewrite the QG vorticity equation in pressure coordinates, and re-dimensionalize. The geostrophic vorticity is

$$\zeta_g = \frac{1}{f_0}\nabla^2\phi$$

but here ϕ is the scaled pressure perturbation (recall from Anelastic equation), and ∇ is taken along surfaces of constant pressure. Substituting in, we have

$$\frac{1}{f_0}\frac{\partial}{\partial t}\nabla^2\phi = -\vec{u}_g \cdot \nabla\left(\frac{1}{f_0}\nabla^2\phi\right) - \beta v_g - f_0\left(\nabla \cdot \vec{u}_a\right)$$

From continuity (EQ. 2), and since the geostrophic wind is non-divergent, $\nabla \cdot \vec{u}_a = -\partial\omega/\partial p$, where ω is the $O(\mathrm{Ro})$ pressure vertical velocity (the leading-order ω_0 vanishes, as before). Thus:

$$\boxed{\frac{1}{f_0}\frac{\partial}{\partial t}\nabla^2\phi = -\vec{u}_g \cdot \nabla\left(\frac{1}{f_0}\nabla^2\phi\right) - \beta v_g + f_0\frac{\partial\omega}{\partial p}} \tag{3}$$

Here, it becomes clear that vortex stretching ($\partial\omega/\partial p > 0$) increases the local rate of vorticity generation. Now we need to eliminate figure out how to write ω in terms of ϕ and/or the geostrophic velocities, which will complete specification of the system. Doing this requires the use of the thermodynamic equation.

The Thermodynamic Equation in Pressure Coordinates

We previously derived the thermodynamic equation under the anelastic approximation in z-coordinates:

$$\frac{D_H b}{Dt} + wN^2 = 0 \tag{4}$$

where

$$\frac{g}{\bar{\theta}}\frac{\partial\bar{\theta}}{\partial z} = N^2 \approx 10^{-4}\ \mathrm{s}^{-2},$$

In pressure coordinates, the thermodynamic equation has a similar form:

$$\frac{D_H T}{Dt} - S_p\,\omega = 0 \tag{5}$$

where S_p is the static stability parameter in pressure coordinates:

$$S_p = -\frac{T}{\theta}\frac{\partial\theta}{\partial p}$$

Since θ increases with height in a stably stratified atmosphere ($\partial\theta/\partial p < 0$), $S_p > 0$.

As we observed earlier, we need to write ω as a function of ϕ in order to close the system. Under the quasi-geostrophic approximation, only the geostrophic wind advects, such that we can expand Eq. (5):

$$\frac{\partial T}{\partial t} + u_g\frac{\partial T}{\partial x} + v_g\frac{\partial T}{\partial y} - S_p\,\omega_1 = 0$$

Since we want to write the above in terms of ϕ, we need a relationship between T and ϕ. This is obtained from hydrostatic balance (in pressure coordinates),

$$\frac{\partial \phi}{\partial p} = -\frac{RT}{p} \qquad \Longrightarrow \qquad T = -\frac{p}{R}\frac{\partial \phi}{\partial p}$$

Substituting,

$$-\frac{p}{R}\frac{\partial}{\partial t}\frac{\partial \phi}{\partial p} - \frac{p}{R}u_g\frac{\partial}{\partial x}\frac{\partial \phi}{\partial p} - \frac{p}{R}v_g\frac{\partial}{\partial y}\frac{\partial \phi}{\partial p} - S_p\,\omega_1 = 0$$

where p can be brought outside of the t, x, and y derivatives because we are in pressure coordinates. Multiplying through by $-R/p$ and defining

$$\sigma \equiv \frac{R\,S_p}{p} = -\frac{RT}{p}\frac{\partial \ln\theta}{\partial p},$$

we have

$$\frac{\partial}{\partial t}\frac{\partial \phi}{\partial p} + u_g\frac{\partial}{\partial x}\frac{\partial \phi}{\partial p} + v_g\frac{\partial}{\partial y}\frac{\partial \phi}{\partial p} + \sigma\,\omega = 0$$

or,

$$\boxed{\frac{\partial}{\partial t}\left(\frac{\partial \phi}{\partial p}\right) + \vec{u}_g \cdot \nabla\left(\frac{\partial \phi}{\partial p}\right) + \sigma\,\omega = 0} \tag{6}$$

This is the quasi-geostrophic temperature equation in pressure coordinates. As we can see here, we have specified ω as a complete function of both ϕ and $\vec{u}_g$!

The Quasi-Geostrophic Geopotential Tendency Equation

Now, we can combine Eq. (3) and (6), to obtain a single equation that tells us how the atmosphere evolves in time. In particular, we will need to take the derivative in pressure of Eq. (6) to obtain a $\partial\omega/\partial p$ term.

Treating σ as a constant (is this a valid assumption?), we multiply Eq. (6) by f_0/σ and take $\partial/\partial p$:

$$\frac{\partial}{\partial p}\left(\frac{f_0}{\sigma}\frac{\partial^2 \phi}{\partial p \partial t} + \frac{f_0}{\sigma}u_g\frac{\partial}{\partial x}\frac{\partial \phi}{\partial p} + \frac{f_0}{\sigma}v_g\frac{\partial}{\partial y}\frac{\partial \phi}{\partial p} + f_0\,\omega\right) = 0$$

or rearranging:

$$\boxed{\frac{f_0}{\sigma}\frac{\partial^2}{\partial p^2}\left(\frac{\partial \phi}{\partial t}\right) = -\frac{f_0}{\sigma}\frac{\partial}{\partial p}\left(u_g\frac{\partial}{\partial x}\frac{\partial \phi}{\partial p} + v_g\frac{\partial}{\partial y}\frac{\partial \phi}{\partial p}\right) - f_0\frac{\partial \omega}{\partial p}}$$

From before, the quasi-geostrophic vorticity equation in pressure coordinates is (Eq. 3):

$$\boxed{\frac{1}{f_0}\frac{\partial}{\partial t}\nabla^2\phi = -\vec{u}_g \cdot \nabla\left(\frac{1}{f_0}\nabla^2\phi\right) - \beta v_g + f_0\frac{\partial \omega}{\partial p}}$$

We observe that both equations contain $f_0\,\partial\omega/\partial p$, so we can **eliminate** ω. From the thermodynamic equation, we write

$$f_0\frac{\partial \omega}{\partial p} = -\frac{f_0}{\sigma}\frac{\partial}{\partial t}\frac{\partial^2 \phi}{\partial p^2} - \frac{f_0}{\sigma}\frac{\partial}{\partial p}\left(\vec{u}_g \cdot \nabla\frac{\partial \phi}{\partial p}\right)$$

Substituting into the vorticity equation:

$$\frac{1}{f_0}\frac{\partial}{\partial t}\nabla^2\phi = -\vec{u}_g \cdot \nabla\left(\frac{1}{f_0}\nabla^2\phi\right) - \beta v_g - \frac{f_0}{\sigma}\frac{\partial}{\partial t}\frac{\partial^2\phi}{\partial p^2} - \frac{f_0}{\sigma}\frac{\partial}{\partial p}\left(\vec{u}_g \cdot \nabla\frac{\partial\phi}{\partial p}\right)$$

Multiplying by f_0, collecting the time derivatives on the left-hand side, and combining the relative and planetary vorticity advection as in the last lecture ($\vec{u}_g \cdot \nabla f = \beta v_g$), we arrive at the **quasi-geostrophic geopotential tendency equation**:

$$\underbrace{\left(\nabla^2 + \frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\frac{\partial\phi}{\partial t}}_{\text{(1) local geopotential tendency}} = \underbrace{-f_0\,\vec{u}_g \cdot \nabla\left(\frac{1}{f_0}\nabla^2\phi + f\right)}_{\text{(2) vorticity advection}} \underbrace{- \frac{f_0^2}{\sigma}\frac{\partial}{\partial p}\left(-\vec{u}_g \cdot \nabla\left(-\frac{\partial\phi}{\partial p}\right)\right)}_{\text{(3) differential temperature advection}} \tag{7}$$

This equation is closed: given ϕ at some instant (from which $\vec{u}_g$ follows), we can solve for the geopotential tendency $\partial\phi/\partial t$, without needing to know ω or $\vec{u}_a$! Let's go through each term to understand what this equation means.

Interpreting the Geopotential Tendency Equation

To make things easier for ourselves, let $\chi \equiv \partial\phi/\partial t$ denote the geopotential tendency.

Term (1): Local Geopotential Tendency

Term (1) involves second derivatives of χ in space. For wave-like patterns, the second derivative of a sinusoid is proportional to minus itself, so

$$\left(\nabla^2 + \frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\chi \;\propto\; -\chi \tag{8}$$

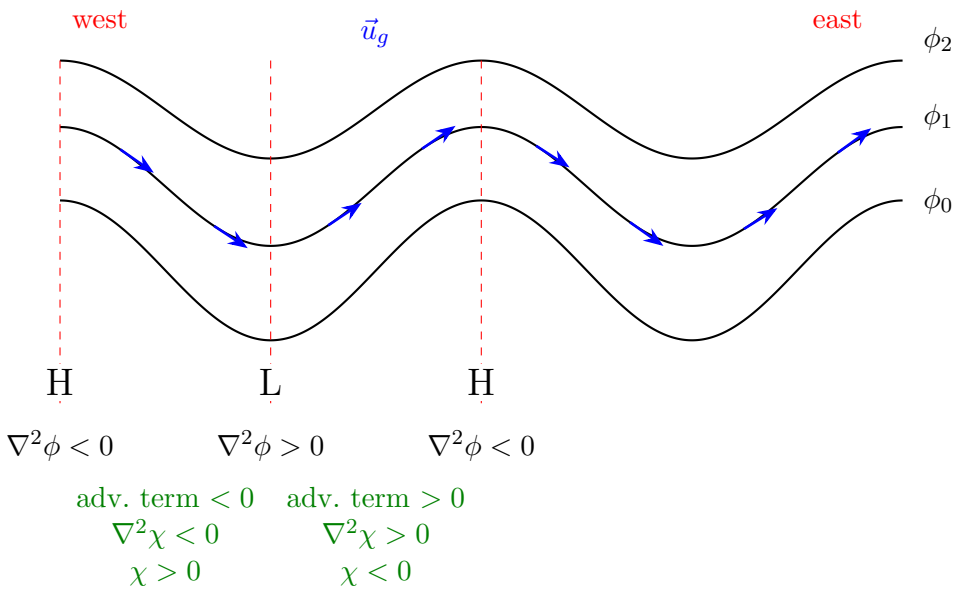
Thus, where the right-hand-side forcing is **positive**, $\chi < 0$ and geopotential heights **fall**; where the forcing is **negative**, $\chi > 0$ and heights **rise**.

Term (2): Vorticity Advection

Term (2) is proportional to the advection of (absolute) vorticity. Suppose an idealized wavy state in the 500-hPa geopotential (see figure on the next page), where $\phi_0 > \phi_1 > \phi_2$ and the flow is westerly:

Between the ridge and the downstream trough, the westerly flow blows from negative (anticyclonic) vorticity toward positive (cyclonic) vorticity, so the geostrophic wind advects negative vorticity into the region. The vorticity advection term is negative, which by the argument for term (1) means $\chi > 0$: **heights rise**. Between the trough and the downstream ridge, the flow advects positive vorticity, the vorticity advection term is positive, and $\chi < 0$: **heights fall**. At the trough and ridge axes themselves, the vorticity gradient along the flow vanishes, and there is no vorticity advection.

The net effect is that heights rise upstream of the trough and fall downstream of it, so the trough (and similarly the ridge) moves eastward. Vorticity advection usually does not change the strength of the disturbance; it only propagates it to the east. Overall, our conclusions are the same as those we made from the QG vorticity equation; indeed, this term is no different from the vorticity advection term in the QG vorticity equation.



Term (3): Differential Temperature Advection

The third term is:

$$-\frac{f_0^2}{\sigma}\frac{\partial}{\partial p}\left(-\vec{u}_g\cdot\nabla\left(-\frac{\partial\phi}{\partial p}\right)\right)$$

From hydrostatic balance, we know that $\partial\phi/\partial p \sim -T$, so this term is proportional to:

$$\sim -\frac{f_0^2}{\sigma}\frac{\partial}{\partial p}\underbrace{(-\vec{u}_g\cdot\nabla T)}_{\text{temperature advection}}$$

Therefore, the third term is proportional to the **change of temperature advection with pressure**, otherwise known as the **differential temperature advection**! Using Eq. (8) again yields:

$$\chi \sim \frac{\partial}{\partial p}(-\vec{u}_g\cdot\nabla T)$$

which tells us that when temperature advection **increases with decreasing pressure** (i.e., the warm advection gets stronger, or the cold advection gets weaker, as we go up in the atmosphere),

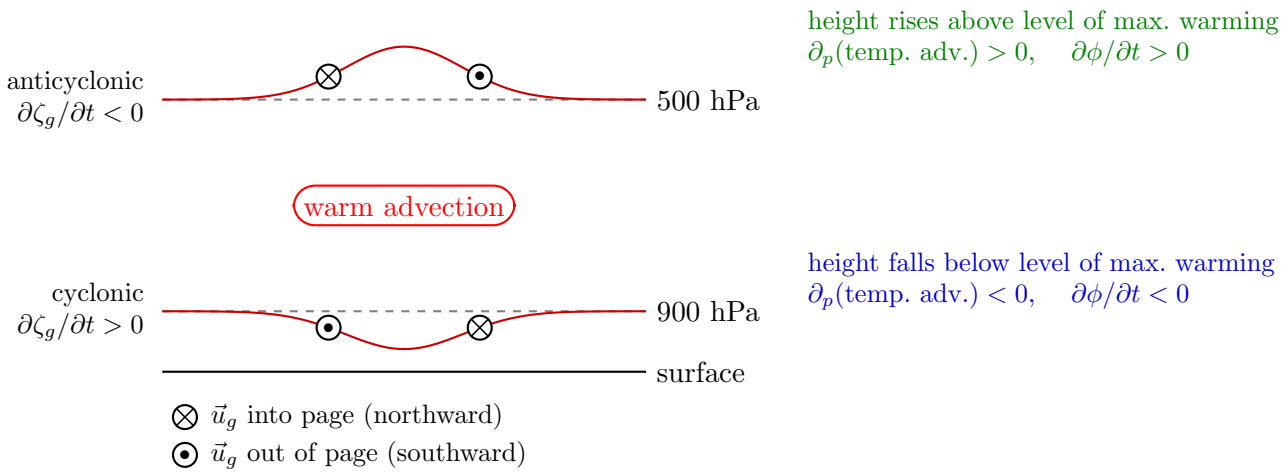
$$\frac{\partial}{\partial p}(-\vec{u}_g\cdot\nabla T) < 0 \qquad \Longrightarrow \qquad \chi < 0 \quad \text{(geopotential height falls)}$$

Conversely, when temperature advection **decreases** with decreasing pressure (the warm advection gets weaker, or the cold advection gets stronger, as we go up in the atmosphere),

$$\frac{\partial}{\partial p}(-\vec{u}_g\cdot\nabla T) > 0 \qquad \Longrightarrow \qquad \chi > 0 \quad \text{(geopotential height rises)}$$

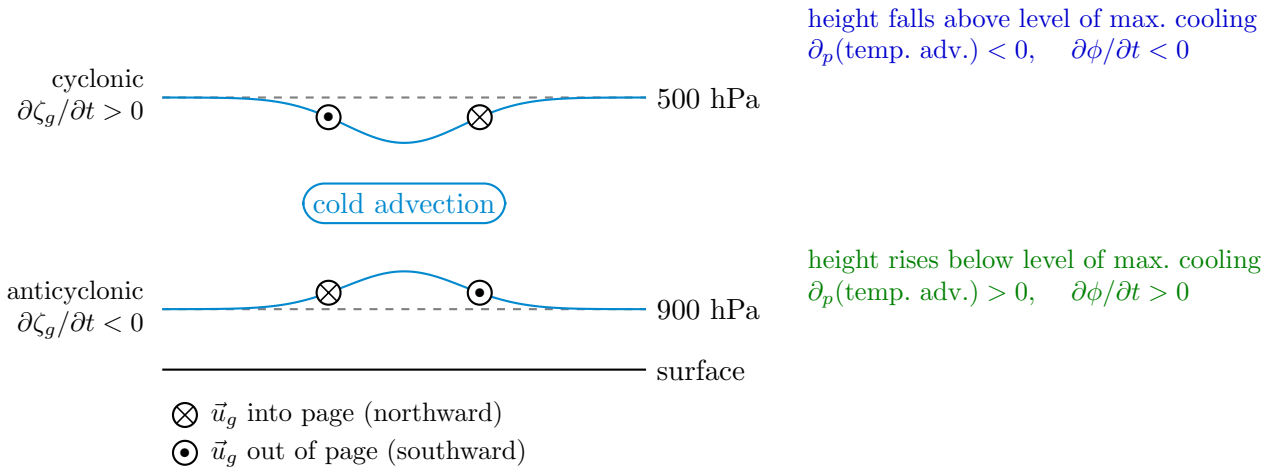
Notice that if the temperature advection is the same at all levels, no matter how strong, this term does not change the geopotential at all. It is not the temperature advection itself that matters, but how it varies in the vertical. The differential temperature advection term plays the role of the vorticity stretching term in the quasi-geostrophic vorticity equation, and we will undertake a series of arguments to exactly understand why.

Consider a layer of warm advection between 500 hPa and 1000 hPa that is **localized in both the vertical and the horizontal**: the warm advection is strongest in the middle of the layer (a maximum in p) and at the center of the diagram (a maximum in x), weakening in all directions away from that point:



Above the level of maximum warm advection, temperature advection decreases with decreasing pressure, so $\partial(\text{temp. adv.})/\partial p > 0$, and heights rise. Below the level of maximum temperature advection, temperature advection increases with decreasing pressure, so $\partial(\text{temp. adv.})/\partial p < 0$ and heights fall.

In contrast, cold advection has the opposite effect. Above the level of maximum cold advection, temperature advection increases with decreasing pressure, so $\partial(\text{temp. adv.})/\partial p < 0$, and heights fall. Below the level of maximum cold advection, temperature advection decreases with decreasing pressure, so $\partial(\text{temp. adv.})/\partial p > 0$ and heights rise.



Why is this related to vorticity stretching? As one can see in the diagrams above, a maximum of warm advection in the middle-layer pushes the heights outwards in both directions: heights rise above the level of maximum warming and fall below it, so the column "stretches" in the vertical direction. In contrast, when there is a maximum of cold advection in the middle-layer, the heights

contract inwards in both directions: heights fall above the level of maximum cooling and rise below it, so the column "compresses" in the vertical direction.

Consider the case where warm advection maximizes in the middle layer. By geostrophic balance ($\zeta_g = \nabla^2\phi/f_0$), we know that falling heights below that temperature advection maxima must be associated with an increase in the *cyclonic vorticity* at that level, while rising heights above the temperature advection maxima must be associated with an increase in the *anticyclonic vorticity* at that level. Since vorticity stretching is associated with increasing cyclonic vorticity, while vorticity compression is associated with increasing anticyclonic vorticity, the increase in cyclonic vorticity at 900-hPa must be associated with vortex stretching, while the increase in anti-cyclonic vorticity at 500-hPa must be associated with vortex compression. This suggests that ω must change as a function of p throughout the column. We will revisit this argument more quantitatively in the future. (Can you figure out the hole in this argument?)

A Simple Intuitive Argument

Recall Charles's Law: the volume of a gas is proportional to its temperature at constant pressure,

$$\frac{V_1}{T_1} = \frac{V_2}{T_2}$$

If a gas **heats up**, it **expands**; if it **cools down**, it **contracts**. If the heating (cooling) is next to the surface, the surface cannot move, so the isobaric surfaces above rise (fall):



Note: it is not the sign of the temperature advection that matters, but its derivative with respect to pressure!

The Quasi-Geostrophic Geopotential Tendency Equation

$$\underbrace{\left(\nabla^2 + \frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\chi}_{\text{local geopotential tendency}} = \underbrace{-f_0\,\vec{u}_g \cdot \nabla\left(\frac{1}{f_0}\nabla^2\phi + f\right)}_{\text{vorticity advection}} \underbrace{-\frac{f_0^2}{\sigma}\frac{\partial}{\partial p}\left(-\vec{u}_g \cdot \nabla\left(-\frac{\partial\phi}{\partial p}\right)\right)}_{\text{differential temperature advection}}$$

where $\chi = \partial\phi/\partial t$ is the geopotential tendency, $\sigma = -\dfrac{RT}{p}\dfrac{\partial \ln\theta}{\partial p}$ is the static stability parameter, and $f = f_0 + \beta y$.

The Quasi-Geostrophic Omega Equation

Chapter 8

Last Modified: November 15, 2026

Atmospheric Vertical Motion

So far, we have talked about the geostrophic winds (u_g, v_g), the geopotential (ϕ), and the geostrophic vorticity (ζ_g). But what about the **vertical velocity**?

The vertical velocity is a very important variable. **Upward** motion is associated with clouds and precipitation, while **downward** motion is associated with cloudlessness, clear skies, and dry conditions. Therefore, vertical motion is one of the most consequential components of a synoptic-scale storm.

However, recall that w is around three orders of magnitude smaller than u and v, which makes it very hard to observe directly. Therefore, it would be great if we could instead infer w from the relationship between u, v, and ϕ. Our goal for this section is to derive and understand the **omega equation**, an equation that allows us to determine the vertical velocity of the atmosphere from its three-dimensional state.[1]

Before we do this, it behooves us to return to how we arrived at the quasi-geostrophic geopotential height tendency equations. Recall the quasi-geostrophic vorticity equation (in pressure coordinates):

$$\boxed{\frac{1}{f_0}\nabla^2\left(\frac{\partial\phi}{\partial t}\right) = -\vec{u}_g\cdot\nabla\left(\frac{1}{f_0}\nabla^2\phi\right) - \beta v_g + f_0\frac{\partial\omega}{\partial p}} \tag{1}$$

as well as the quasi-geostrophic temperature equation (in pressure coordinates):

$$\boxed{\frac{\partial}{\partial p}\left(\frac{\partial\phi}{\partial t}\right) + \vec{u}_g\cdot\nabla\left(\frac{\partial\phi}{\partial p}\right) + \sigma\,\omega = 0} \tag{2}$$

where in both equations we have exchanged the order of the spatial and temporal derivatives. In order to combine these two equations, we noticed that they both included ω. Thus, we eliminated ω to get a single equation that told us how ϕ evolves in time (the quasi-geostrophic geopotential height tendency equation). However, eliminating ω means that we lose information about the vertical velocity.

Notice that both equations also contain the time tendency of ϕ. Instead of eliminating ω, what if we instead eliminated the **time derivative** term, $\partial\phi/\partial t$? In theory, this should also give an equation that allows us to calculate ω given a bunch of other terms. Indeed, we will proceed with this, and see what we obtain!

[1] The name comes from the pressure-coordinate version of this equation, which is written for $\omega = Dp/Dt$.

Deriving the Quasi-Geostrophic Omega Equation

For convenience, let $\chi \equiv \partial\phi/\partial t$ denote the geopotential tendency, as before. Our strategy will be to manipulate each equation so that the time-tendency terms become identical, and then subtract one equation from the other.

Step 1: The Vorticity Equation

Multiplying the QG vorticity equation (Eq. 1) by f_0:

$$\nabla^2\chi = -f_0\,\vec{u}_g \cdot \nabla\zeta_g - f_0\beta v_g + f_0^2\frac{\partial\omega}{\partial p}$$

Taking $\partial/\partial p$ of this equation, and noting that $\partial/\partial p$ commutes with ∇^2 (since ∇ is taken along surfaces of constant pressure),

$$\nabla^2\left(\frac{\partial\chi}{\partial p}\right) = f_0\frac{\partial}{\partial p}\left(-\vec{u}_g \cdot \nabla\zeta_g\right) - f_0\beta\frac{\partial v_g}{\partial p} + f_0^2\frac{\partial^2\omega}{\partial p^2} \tag{3}$$

Step 2: The Thermodynamic Equation

Substituting for χ, the quasi-geostrophic temperature equation (Eq. 2) becomes:

$$\frac{\partial\chi}{\partial p} + \vec{u}_g \cdot \nabla\left(\frac{\partial\phi}{\partial p}\right) + \sigma\,\omega = 0$$

Taking the horizontal Laplacian of this equation, and noting that $\sigma = \sigma(p)$ only, so that it can be brought outside of ∇^2,

$$\nabla^2\left(\frac{\partial\chi}{\partial p}\right) = -\nabla^2\left[\vec{u}_g \cdot \nabla\left(\frac{\partial\phi}{\partial p}\right)\right] - \sigma\nabla^2\omega \tag{4}$$

Step 3: Eliminate the Time Tendency

Notice that the left-hand sides of Eq. (3) and Eq. (4) are identical! Setting the right-hand sides equal,

$$-\nabla^2\left[\vec{u}_g \cdot \nabla\left(\frac{\partial\phi}{\partial p}\right)\right] - \sigma\nabla^2\omega = f_0\frac{\partial}{\partial p}\left(-\vec{u}_g \cdot \nabla\zeta_g\right) - f_0\beta\frac{\partial v_g}{\partial p} + f_0^2\frac{\partial^2\omega}{\partial p^2}$$

Collecting the terms in ω on the left-hand side and dividing by σ, we arrive at the **quasi-geostrophic omega equation**:

$$\boxed{\left(\nabla^2 + \frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega = \underbrace{-\frac{f_0}{\sigma}\frac{\partial}{\partial p}\left(-\vec{u}_g \cdot \nabla\zeta_g\right)}_{S_1} + \underbrace{\frac{f_0}{\sigma}\frac{\partial(\beta v_g)}{\partial p}}_{S_2} \underbrace{-\frac{1}{\sigma}\nabla^2\left[-\vec{u}_g \cdot \nabla\left(-\frac{\partial\phi}{\partial p}\right)\right]}_{S_3}} \tag{5}$$

Notice that the operator on the left-hand side is **exactly the same** as the operator acting on χ in the geopotential tendency equation! Both equations come from the same pair of balanced equations; we have simply eliminated a different variable. Also notice that the omega equation contains **no time derivatives**. It is purely a *diagnostic* equation, which tells us the vertical motion pattern given the instantaneous state of the atmosphere. In the next section, we will interpret the quasi-geostrophic ω equation.

Interpretation

Wow, this equation looks very complicated! Let's try to break it down. Mathematically, the first term S_1 is the pressure derivative of the relative vorticity advection, or the **differential relative vorticity advection**. The second term S_2 is the pressure derivative of the planetary vorticity advection, or the **differential planetary vorticity advection**. The third term S_3 is proportional to the Laplacian of temperature advection. This is because we can use hydrostatic balance, $\partial\phi/\partial p = -RT/p$, to rewrite the thermal advection term in terms of temperature,

$$S_3 = -\frac{1}{\sigma}\nabla^2\left[-\vec{u}_g \cdot \nabla\left(-\frac{\partial\phi}{\partial p}\right)\right] = -\frac{R}{\sigma p}\nabla^2\left(-\vec{u}_g \cdot \nabla T\right)$$

You might also notice that the two vorticity advection terms can be combined into a single differential *absolute* vorticity advection term, since $\vec{u}_g \cdot \nabla f = \beta v_g$:

$$S_1 + S_2 = -\frac{f_0}{\sigma}\frac{\partial}{\partial p}\left[-\vec{u}_g \cdot \nabla\left(\zeta_g + f\right)\right]$$

Thermal Wind Balance and Quasi-Geostrophic Omega

Before we analyze each individual term, we must first understand the physical meaning of deriving the quasi-geostrophic omega equation. Recall Equations 3 and 4. We set the left-hand-sides equal to each other, but what exactly do the left-hand sides mean?

To understand this better, recall that the geostrophic vorticity is related to the geopotential by

$$\zeta_g = \frac{1}{f_0}\nabla^2\phi \qquad \Longrightarrow \qquad f_0\zeta_g = \nabla^2\phi$$

where ∇ is taken along surfaces of constant pressure. Taking the derivative with respect to p of both sides, and noting that $\partial/\partial p$ commutes with the horizontal Laplacian on constant pressure surfaces,

$$f_0\frac{\partial\zeta_g}{\partial p} = \frac{\partial}{\partial p}\left(\nabla^2\phi\right) = \nabla^2\left(\frac{\partial\phi}{\partial p}\right)$$

From hydrostatic balance (in pressure coordinates), the vertical derivative of the geopotential is related to the temperature,

$$\frac{\partial\phi}{\partial p} = -\frac{RT}{p}$$

so that substituting, we obtain **thermal wind balance**, except in terms of the vorticity:

$$\boxed{f_0\frac{\partial\zeta_g}{\partial p} = \nabla^2\left(\frac{\partial\phi}{\partial p}\right) = -\frac{R}{p}\nabla^2 T} \tag{6}$$

where p can be brought outside of ∇^2 because we are in pressure coordinates. Just like the thermal wind balance equation for the geostrophic wind, this version of thermal wind balance relates the **vertical shear** of the geostrophic vorticity to the **horizontal gradients** (or, the Laplacian) of temperature! A local temperature maximum ($\nabla^2 T < 0$) requires $\partial\zeta_g/\partial p > 0$, i.e., the vorticity must *decrease* with height (become more anticyclonic aloft and more cyclonic below), consistent with the warm-core picture from the geopotential tendency lecture. In contrast, with a local temperature minimum ($\nabla^2 T > 0$), the vorticity must *increase* (become more cyclonic) with height. In other words, **warm-core** cyclones have vorticity decreasing with height, while **cold-core** cyclones have vorticity increasing with height.

Let us next take the time-derivative of Equation (6), for reasons which will become clear shortly:

$$\boxed{f_0 \frac{\partial}{\partial t}\left(\frac{\partial \zeta_g}{\partial p}\right) = -\frac{R}{p}\frac{\partial}{\partial t}(\nabla^2 T)} \tag{7}$$

This equation states that if the atmosphere is to **remain** in thermal wind balance as it evolves, then any change in the vertical shear of the vorticity (left-hand side) must be accompanied by a corresponding change in the horizontal Laplacian of the temperature (right-hand side). Notice that the left-hand side involves the time tendency of the **vorticity**, which is governed by the QG vorticity equation, while the right-hand side involves the time tendency of the **temperature**, which is governed by the quasi-geostrophic thermodynamic equation. Thermal wind balance therefore links the two equations together. Furthermore, notice that the left-hand-side of Equation 7 is equal to:

$$f_0 \frac{\partial}{\partial t}\left(\frac{\partial \zeta_g}{\partial p}\right) = \frac{\partial}{\partial t}\left[\nabla^2\left(\frac{\partial \phi}{\partial p}\right)\right] = \nabla^2\left[\frac{\partial}{\partial p}\left(\frac{\partial \phi}{\partial t}\right)\right] = \nabla^2\left(\frac{\partial \chi}{\partial p}\right)$$

which is exactly equal to the left-hand-sides of Equations 3 and 4. So their two left-hand sides are actually **the time tendencies of the two sides of thermal wind balance**! Therefore, requiring the left-hand sides to be equal is the same as requiring the flow to **remain in exact thermal wind balance as it evolves in time**. We will expand on this more later.

Interpreting the Source Terms

So how do we interpret the quasi-geostrophic omega equation? Let us step through each term individually. Recall that, since the operator on the left-hand side is related to the second-derivative, ω is **anticorrelated** with the source terms. Since upward motion corresponds to $\omega < 0$ in pressure coordinates, then

$$S > 0 \implies \omega < 0 \text{ (ascent)}, \qquad S < 0 \implies \omega > 0 \text{ (descent)}$$

S_1: Differential Vorticity Advection

If the vorticity advection, $-\vec{u}_g \cdot \nabla \zeta_g$, **increases with height** (i.e., increases with *decreasing* pressure), then

$$\frac{\partial}{\partial p}\left(-\vec{u}_g \cdot \nabla \zeta_g\right) < 0 \quad \rightarrow \quad \frac{\partial}{\partial p}\left(\vec{u}_g \cdot \nabla \zeta_g\right) > 0 \quad \rightarrow \quad S_1 > 0 \quad \rightarrow \quad \omega < 0 \quad \text{(ascent!)}$$

In other words,

- **cyclonic** vorticity advection *increasing* with height, or **anticyclonic** vorticity advection *decreasing* with height leads to $\omega < 0$, or **ascent**
- **cyclonic** vorticity advection *decreasing* with height, or **anticyclonic** vorticity advection *increasing* with height, leads to $\omega > 0$, or **descent**

S_2: Differential Planetary Vorticity Advection

S_2 is also a differential vorticity advection term, but of the *planetary* vorticity, since $\vec{u}_g \cdot \nabla f = \beta v_g$. Its interpretation is the same as S_1; when planetary vorticity advection increases with height, there must be ascent, and when planetary vorticity advection decreases with height, there must be descent.

S_3: Thermal Advection

Recall that the thermal advection term can be written as

$$S_3 = -\frac{R}{\sigma p}\nabla^2\left(-\vec{u}_g \cdot \nabla T\right)$$

At a local maximum of **warm** advection, $-\vec{u}_g \cdot \nabla T$ has a local maximum, so

$$\nabla^2\left(-\vec{u}_g \cdot \nabla T\right) < 0 \qquad \rightarrow \qquad S_3 > 0 \qquad \rightarrow \qquad \omega < 0 \quad \text{(ascent)}$$

Similarly, a local maximum of **cold** advection leads to $\omega > 0$ (**descent**).

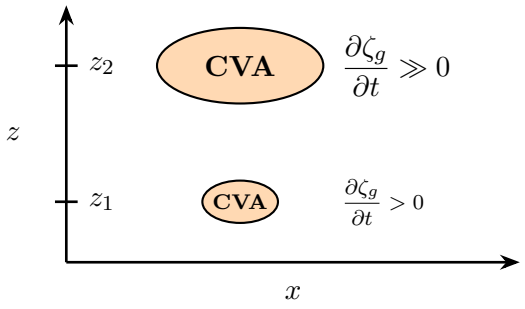
We could, very easily, stop here and leave you to memorize these relationships. But it is much more instructive to interpret the meaning of this. As we shall see shortly, this will give us deep insight into the behavior of the atmosphere on synoptic scales.

Thermal Wind Balance and Geostrophic Advection

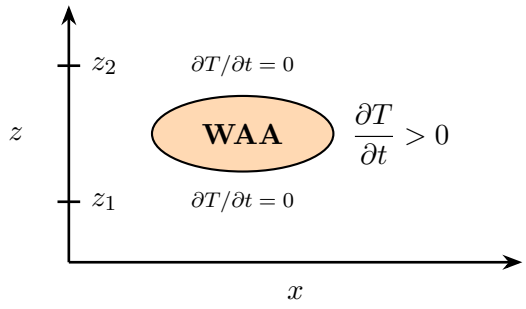
To build intuition, it is easier to think in terms of height rather than pressure. Since pressure decreases with height, thermal wind balance (Eq. 6) can equivalently be written as

$$\frac{\partial u_g}{\partial z} \propto -\frac{\partial T}{\partial y} \qquad \text{and} \qquad f_0 \frac{\partial \zeta_g}{\partial z} \propto \nabla^2 T$$

i.e., the vertical shear of the geostrophic wind (vorticity) must be consistent with the horizontal temperature gradient (Laplacian). Let us now ask what advection by the geostrophic wind does to each side of this balance. We consider two cases, in an x–z cross-section with two levels $z_1 < z_2$.



Case 1



Case 2

Case 1: Differential Vorticity Advection

Suppose there is cyclonic vorticity advection (CVA) at both z_1 and z_2, but it is stronger at z_2, i.e., the CVA **increases with height**. According to the QG vorticity equation,

$$\frac{\partial \zeta_g}{\partial t} = -\vec{u}_g \cdot \nabla\left(\zeta_g + f\right) - f_0\left(\nabla \cdot \vec{u}_a\right)$$

and ignoring the ageostrophic term for now, the vorticity increases at both levels ($\partial\zeta_g/\partial t > 0$), but it increases *more* at z_2 than at z_1. Thus, $\partial\zeta_g/\partial z$ **increases**. However, vorticity advection does nothing to the temperature, so $\nabla^2 T$ does not change. **The vertical shear changes, but not the horizontal temperature gradient!** Thermal wind balance is being disrupted.

Case 2: Thermal Advection

Now suppose there is a local maximum of warm air advection (WAA) centered at some level z^* between z_1 and z_2. Importantly, the WAA is **localized in the horizontal**: it is strongest at the center of the diagram and weakens away from it. According to the QG thermodynamic equation,

$$\frac{\partial T}{\partial t} = -\vec{u}_g \cdot \nabla T + S_p \, \omega$$

and ignoring the vertical motion term for now, the temperature increases ($\partial T/\partial t > 0$) most strongly at the center of the WAA, and less so away from it. This creates a local **horizontal** maximum of temperature at z^*, so $\nabla^2 T$ **decreases** there. However, there is no vorticity advection, so $\partial \zeta_g/\partial z$ at z^* does not change. **The horizontal temperature gradient changes, but not the vertical shear of vorticity!** Thermal wind balance is being disrupted.

A central point being made here is that in both of these cases, advection of either vorticity or temperature by the geostrophic wind is destroying thermal wind balance! This is quite insightful, as it means that the primary balance in the atmosphere is continuously disrupting itself.

Restoring Thermal Wind Balance

If geostrophic balance is constantly disrupting thermal wind balance, why do we see an atmosphere that is in thermal wind balance? One way that this could be true is if thermal wind balance is somehow restored by another process. In fact, as we saw earlier, when deriving the quasi-geostrophic ω equation, we required the flow to remain in exact thermal wind balance in time. Since thermal wind balance relates horizontal temperature gradients to vertical wind shear, we have two choices:

- increase the vertical wind shear
- decrease the horizontal temperature gradient.

Let's first focus on how thermal wind balance is restored through modification of the horizontal temperature gradient. Looking at the quasi-geostrophic temperature equation (Equation 2), the only other way to change temperature, besides advection, is through the vertical velocity. Let us write the quasi-geostrophic temperature equation in terms of T directly:

$$\frac{\partial T}{\partial t} = -\vec{u}_g \cdot \nabla T + S_p \, \omega \tag{8}$$

where $S_p > 0$ for a stably stratified atmosphere. If the advection term is zero, then:

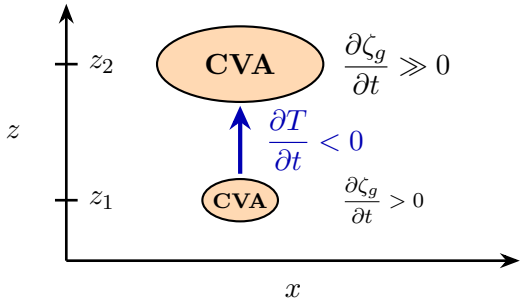
- when $\omega < 0$, a parcel **cools**, since it moves to lower pressure and expands adiabatically
- when $\omega > 0$, a parcel **warms**, since it moves to higher pressure and compresses adiabatically

It turns out that the vertical motion from the QG omega equation is **exactly** that which is needed to restore thermal wind balance! Let us revisit our two cases.

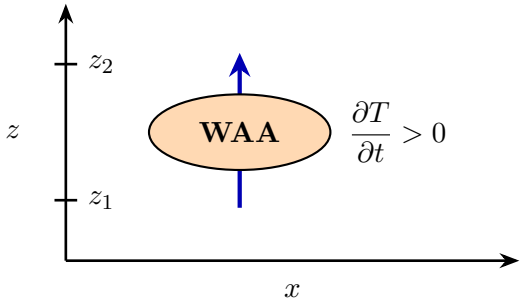
Case 1 Revisited: Differential Vorticity Advection

The cyclonic vorticity advection increasing with height has increased $\partial \zeta_g/\partial z$. To restore thermal wind balance, $\nabla^2 T$ must also increase, which means we need to generate a local **minimum** of temperature between z_1 and z_2. This requires cooling, i.e., **upward motion**, so that $\partial T/\partial t < 0$ between the two levels.

This is exactly what the QG omega equation predicts: cyclonic vorticity advection increasing with height means $S_1 > 0$, and thus $\omega < 0$ (**ascent**), helping to restore thermal wind balance!



ascent cools: $\nabla^2 T$ increases



ascent cools, offsetting the WAA

Case 2 Revisited: Thermal Advection

The local maximum of WAA has decreased $\nabla^2 T$ by generating a local maximum of temperature. To restore balance, we need to cool the air where the warm advection is occurring. Upward motion is associated with adiabatic expansion (cooling), which **offsets** the increase in temperature from the WAA.

Again, this is exactly what the QG omega equation predicts: a local maximum of warm advection means $S_3 > 0$, and thus $\omega < 0$ (**ascent**), helping to restore thermal wind balance!

Connecting to Vorticity Stretching and Compression

The vertical motion does not only change the temperature. By continuity, a vertical motion that varies with height must be accompanied by horizontal convergence and divergence, which change the vorticity through vortex stretching and compression. Recall the vortex stretching term in the QG vorticity equation (Eq. 1),

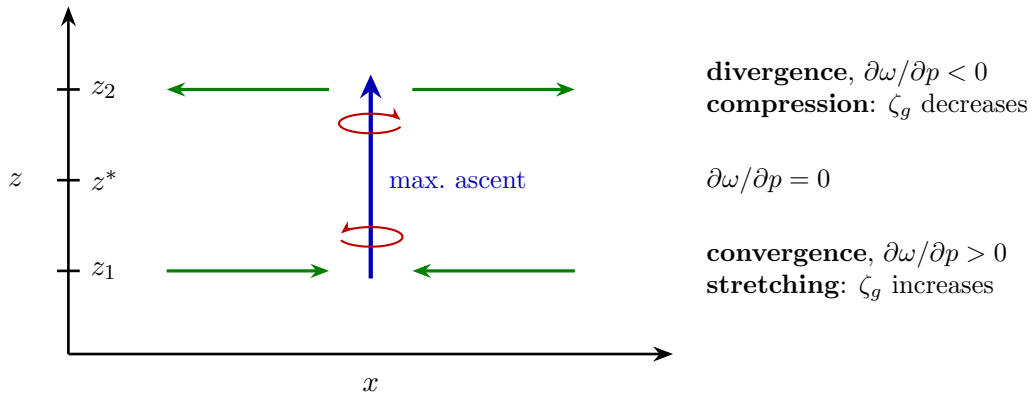
$$\frac{\partial \zeta_g}{\partial t} = \cdots + f_0 \frac{\partial \omega}{\partial p}$$

In both of our cases, the ascent is strongest in the middle of the layer (near z^*) and weaker above and below it. This has consequences for the vorticity tendency at both levels.

Below the level of maximum ascent, the upward motion increases with height, so air must **converge** horizontally to conserve mass. This means that ω becomes more negative with decreasing p, such that $\partial\omega/\partial p > 0$. In other words, the column is **stretched**, and the vorticity **increases**, or becomes more cyclonic.

Above the level of maximum ascent, the upward motion decreases with height, so air must **diverge** horizontally to conserve mass. Here $\partial\omega/\partial p < 0$, the vortex columns are **compressed**, and the vorticity **decreases**, or becomes more anticyclonic.

The net effect is that the vorticity increases below and decreases above, so the ascent **decreases** $\partial\zeta_g/\partial z$. It turns out this is exactly what is needed to help restore thermal wind balance in both cases!



Net effect: $\partial\zeta_g/\partial z$ decreases

In **Case 1**, the CVA increasing with height *increased* $\partial\zeta_g/\partial z$. The vortex stretching below and compression above partially undo this increase, while the adiabatic cooling *increases* $\nabla^2 T$ to match. Together, these two effects bring both sides of the thermal wind balance equation back into agreement.

In **Case 2**, the WAA *decreased* $\nabla^2 T$, so thermal wind balance requires $\partial\zeta_g/\partial z$ to decrease as well. The vortex stretching below and compression above do exactly this, while the adiabatic cooling partially offsets the decrease in $\nabla^2 T$. Again, these two effects bring both sides of the thermal wind balance equation back into agreement.

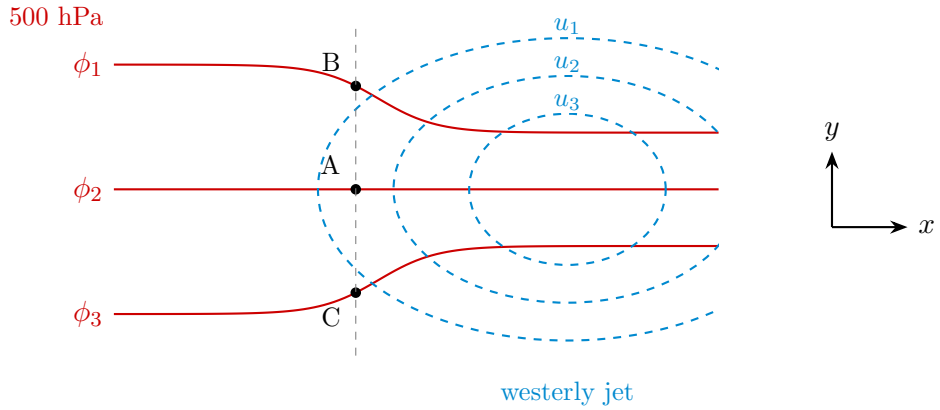
From this analysis, we can clearly see that the **vertical motion restores thermal wind balance** by acting on both sides of the balance at once: it changes the temperature through adiabatic expansion and compression, and it changes the vertical shear of vorticity through vortex stretching and compression. Neither effect alone is sufficient. The quasi-geostrophic omega equation determines precisely the vertical motion for which the sum of these two effects keeps the atmosphere in thermal wind balance at all times. This is also why the vorticity stretching term in the quasi-geostrophic vorticity equation and the differential temperature advection term in the geopotential tendency equation are so closely related, since both describe the vertical motion's role in maintaining thermal wind balance!

Summary

The next time you are outside, and it is precipitating owing to the passage of a synoptic-scale storm, you are actually experiencing the atmosphere's way of **restoring thermal wind balance**! This is because thermal wind balance is continuously destroyed by itself, through advection by the geostrophic wind. Vertical motion is required to restore thermal wind balance by changing the temperature (through adiabatic expansion and compression) and the vertical shear of vorticity (through vortex stretching and compression), so that the two remain consistent.

Jet Streaks and Synoptic-Scale Storm Development

We are now equipped with the ability to analyze jet streaks, which are often associated with the development of synoptic-scale storms. Consider a westerly jet streak at 500 hPa, where the geopotential contours converge as the flow enters the jet. Here $\phi_1 < \phi_2 < \phi_3$, so that $u_g \sim -\partial\phi/\partial y > 0$, and the isotachs (dashed) satisfy $u_1 < u_2 < u_3$:



To simplify, suppose the 1000-hPa geopotential is flat (zero everywhere), so that the 500-hPa geopotential is also the **thickness**, i.e., the contours of ϕ are also isotherms.

At point A, the geostrophic flow is accelerating downstream, so the advection of momentum

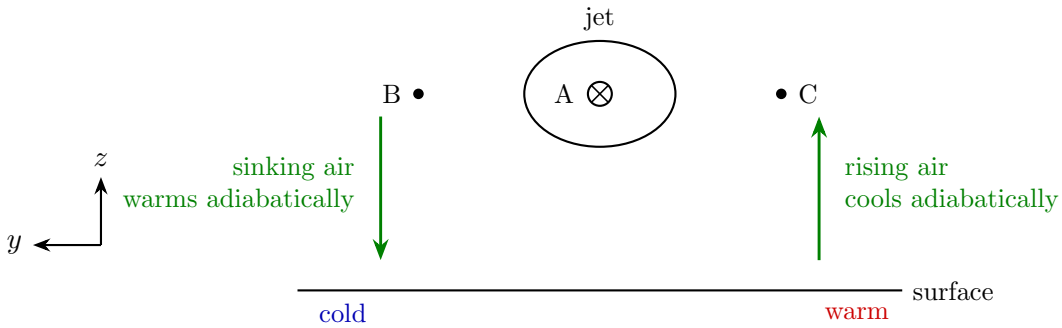
$$-u_g \frac{\partial u_g}{\partial x} < 0$$

reduces u_g at the entrance of the jet. In other words, $\partial u_g/\partial t < 0$ at Point A. On the other hand, the thermal advection is ≈ 0, since $\vec{u}_g$ is parallel to the contours of ϕ (which are the isotherms). Next, recall thermal wind balance:

$$\frac{\partial u_g}{\partial z} \sim -\frac{\partial T}{\partial y}$$

Since $\partial u_g/\partial t < 0$ at Point A (500-hPa) but $u_g = 0$ at the surface, the shear $\partial u_g/\partial z$ **decreases** over time. However, the thermal advection, $\vec{u}_g \cdot \nabla T$, is zero. This means that the meridional temperature gradient does not change. Because the vertical shear is changing, but the meridional temperature gradient is not, thermal wind balance is being disrupted. In other words, we see yet another example where geostrophic advection destroys thermal wind balance.

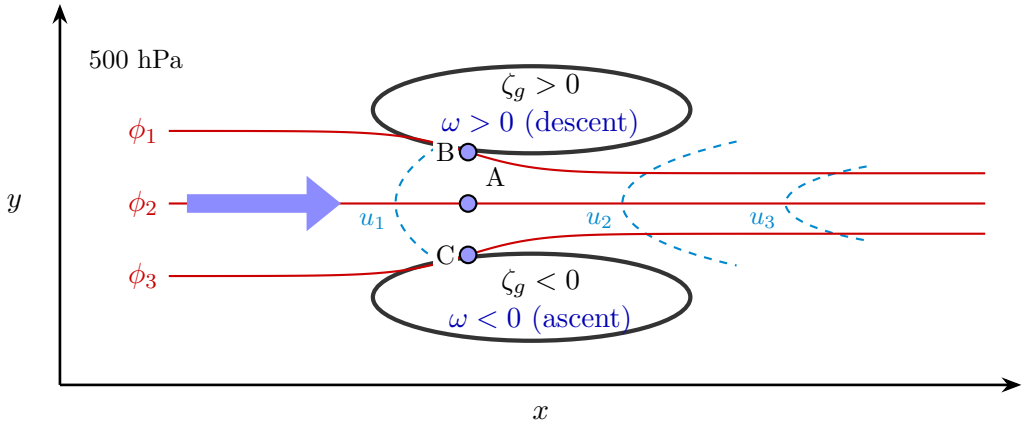
Let us investigate further by looking along a y–z cross-section through B, A, and C, with the jet (into the page) at Point A:



Since momentum advection will decrease the zonal velocity at Point A, eventually the vertical wind shear will become too weak for the existing meridional temperature gradient. We can reduce the meridional temperature gradient by cooling at Point C, and warming at Point B. Looking at the quasi-geostrophic temperature equation (Equation 2), the only other way to change temperature, besides advection, is through the vertical velocity. Therefore, to weaken the meridional temperature gradient and restore thermal wind balance, we need **ascent** on the warm, equatorward side of the jet (Point C), and **descent** on the cold, poleward side of the jet (Point B). The resulting transverse circulation is drawn by the green arrows above. This is called a **thermally direct** circulation, since warm air rises and cold air sinks.

Is This Consistent with the Quasi-Geostrophic Omega Equation?

Let's return to the plan view. North of the jet axis (near B), the vorticity is cyclonic ($\zeta_g > 0$), and it increases downstream as the jet strengthens. The westerly flow therefore advects *lower* vorticity from upstream, i.e., there is **anticyclonic vorticity advection (AVA)** at B. South of the jet axis (near C), the vorticity is anticyclonic ($\zeta_g < 0$) and becomes more negative downstream, so there is **cyclonic vorticity advection (CVA)** at C.



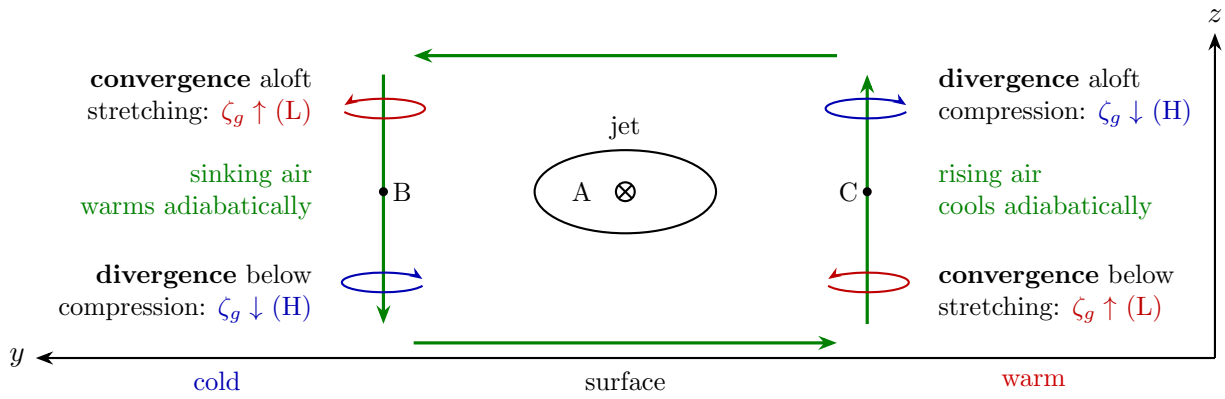
At B, the vorticity advection is negative, such that the vorticity advection decreases with height (from the surface). This means $S_1 < 0$ and $\omega > 0$, or **descent**.

At C, the vorticity advection is positive, such that the vorticity advection increases with height (from the surface). This means $S_1 > 0$ and $\omega < 0$, or **ascent**.

Notice that this is exactly the vertical motion that we deduced was required to restore thermal wind balance! What happens at the surface below Points B and C?

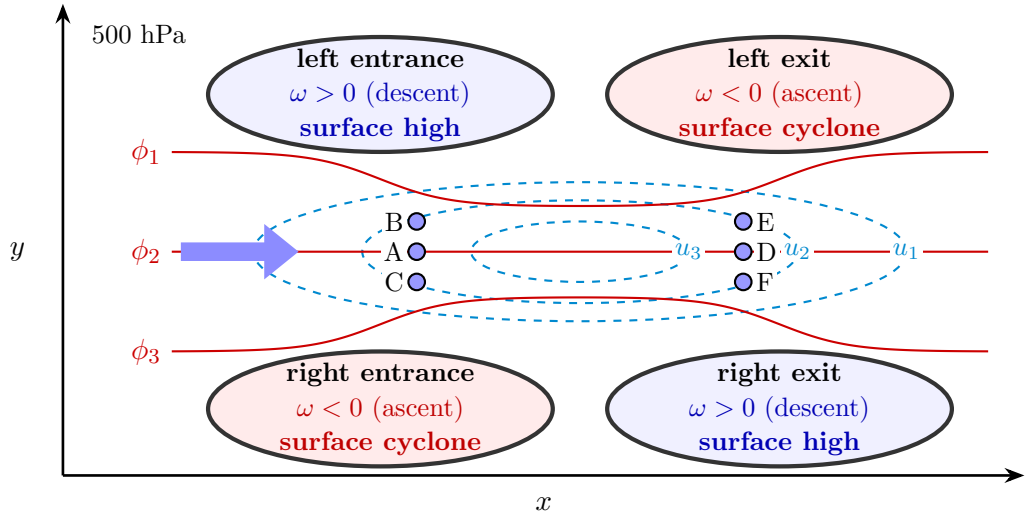
As we learned earlier, a maximum of vertical velocity at mid-levels coincides with vortex stretching near the surface, and vortex compression aloft. In other words, there is convergence and increasing cyclonic vorticity at the surface below Point C, and divergence and increasing anticyclonic vorticity in the atmosphere above Point C. Likewise, there is divergence and increasing anticyclonic vorticity at the surface below Point B, and convergence and increasing cyclonic vorticity in the atmosphere above Point B. Notice the similarity with the diagram on Page 8.

This is why surface cyclones tend to develop beneath Point C, which is also called the right jet entrance, where divergence aloft draws air upward and the resulting surface convergence stretches the column and increases cyclonic vorticity. Notably, it is also the region where cyclonic vorticity advection increases with height, which must have upwards motion to restore thermal wind balance.



In contrast, surface high pressures tend to develop beneath Point B, which is also called the left jet entrance. There, convergence aloft is associated with sinking air, and vortex compression increases surface anticyclonic vorticity.

If we generalize to the jet exit region, the pattern reverses (can you work out why?), as shown in the figure below. Below Point E, surface convergence is associated with mid-level ascent that can be diagnosed from the increasing cyclonic vorticity advection with height. Like the right jet entrance region, surface cyclones tend to develop in the left exit region. In contrast, increasing anticyclonic vorticity advection with height occurs in the right exit region, which also must be associated with descent and surface divergence. Like the left jet entrance region, surface anticyclones tend to develop in the right exit region.



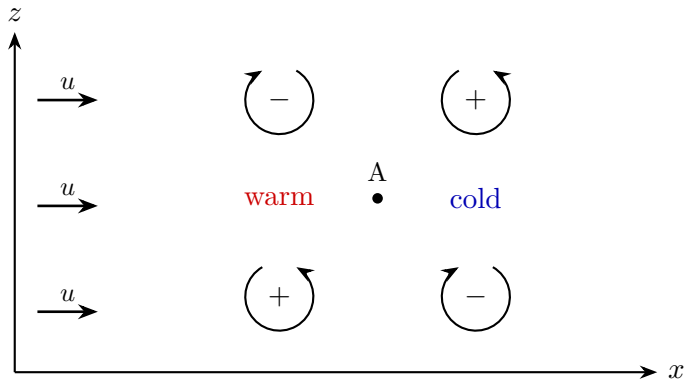
The Quasi-Geostrophic Omega Equation

$$\left[\nabla_p^2 + \frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right]\omega = \underbrace{-\frac{f_0}{\sigma}\frac{\partial}{\partial p}\left(-\vec{u}_g \cdot \nabla_p \zeta_g\right)}_{\text{differential vorticity advection}} + \underbrace{\frac{f_0}{\sigma}\frac{\partial \beta v_g}{\partial p}}_{\text{differential planetary vorticity advection}} \underbrace{-\frac{1}{\sigma}\nabla_p^2\left[-\vec{u}_g \cdot \nabla_p\left(-\frac{\partial \phi}{\partial p}\right)\right]}_{\text{thermal advection}}$$

- Differential vorticity advection and thermal advection (advection by the geostrophic wind) tend to disrupt thermal wind balance
- Vertical motion arises in response, in order to restore thermal wind balance

Issues with the Quasi-Geostrophic Omega Equation

While the quasi-geostrophic omega equation is able to tell us where there is vertical motion given source functions on the right-hand-side of the equation, it turns out that the thermal advection term (S_3) often cancels out with the differential vorticity advection term (S_1). To see this, consider the following x–z cross-section, with a basic zonal flow that has **no dependence on** z, and a warm region upstream of a cold region:



This structure is in thermal wind balance: $f_0\,\partial\zeta_g/\partial z \propto \nabla_H^2 T \sim -T$, so the vorticity decreases with height over the warm region (cyclonic below, anticyclonic above) and increases with height over the cold region.

Next, let us consider both the thermal advection and differential vorticity advection at Point A.

At point A, there is a maximum of **warm advection** (S_3), which will force **ascent** at Point A. Taken alone, the warm advection changes $\nabla_H^2 b$ (the horizontal gradient of temperature) at A, without any change to $\partial\zeta_g/\partial z$. Therefore, ascent is required to reduce the horizontal temperature gradient at Point A and restore thermal wind balance.

At the same time, the vorticity advection is positive below Point A and negative above Point A. The cyclonic vorticity advection thus decreases with height, which is associated with descent.

Therefore, the vertical motions associated with differential vorticity advection and thermal advection are of opposite sign! And, in fact, one can show that they exactly cancel. Why does this make sense? It turns out that this atmosphere is exactly in thermal wind balance, and the advection does **not** disrupt thermal wind balance, since a vertically uniform flow simply translates the whole pattern downstream. Thus, there is no need for any vertical motion to restore thermal wind balance.

Q-Vectors and the Ageostrophic Circulation

Chapter 9

Last Modified: November 15, 2026

Where We Left Off

In the last few lectures, we derived the quasi-geostrophic omega equation in pressure coordinates, and interpreted it as the vertical motion required to keep the atmosphere in thermal wind balance. The quasi-geostrophic omega equation is:

$$\boxed{\left(\nabla^2 + \frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega = \underbrace{\frac{f_0}{\sigma}\frac{\partial}{\partial p}\left(\vec{u}_g \cdot \nabla(\zeta_g + f)\right)}_{S_1, S_2\text{: diff. vorticity advection}} \underbrace{-\frac{1}{\sigma}\nabla^2\left[\vec{u}_g \cdot \nabla\left(\frac{\partial \phi}{\partial p}\right)\right]}_{S_3\text{: thermal advection}}} \tag{1}$$

Recall that since the operator on the left-hand side is elliptic, ω is anticorrelated with the source terms on the right-hand side. $S > 0$ implies ascent ($\omega < 0$), and $S < 0$ implies descent ($\omega > 0$).

The Cancellation Problem

In practice, using Eq. (1) to estimate the sign of ω from a weather map is difficult. To see why, let us expand both source terms above using the chain rule. The differential vorticity advection term is

$$S_1 = \frac{f_0}{\sigma}\frac{\partial \vec{u}_g}{\partial p} \cdot \nabla \zeta_g \; + \; \underline{\frac{f_0}{\sigma}\,\vec{u}_g \cdot \nabla\left(\frac{\partial \zeta_g}{\partial p}\right)}$$

while the thermal advection term is

$$S_3 = -\frac{1}{\sigma}\left(\nabla^2 \vec{u}_g\right) \cdot \nabla\left(\frac{\partial \phi}{\partial p}\right) \; - \; \frac{2}{\sigma}\left[\frac{\partial \vec{u}_g}{\partial x} \cdot \nabla\left(\frac{\partial^2 \phi}{\partial x \partial p}\right) + \frac{\partial \vec{u}_g}{\partial y} \cdot \nabla\left(\frac{\partial^2 \phi}{\partial y \partial p}\right)\right] \; - \; \frac{1}{\sigma}\,\vec{u}_g \cdot \nabla\left[\nabla^2\left(\frac{\partial \phi}{\partial p}\right)\right]$$

Recall from the previous notes that thermal wind balance, written in terms of the vorticity, is

$$f_0 \frac{\partial \zeta_g}{\partial p} = \nabla^2\left(\frac{\partial \phi}{\partial p}\right)$$

Substituting this into the last term of S_3,

$$-\frac{1}{\sigma}\,\vec{u}_g \cdot \nabla\left[\nabla^2\left(\frac{\partial \phi}{\partial p}\right)\right] = \underline{-\frac{f_0}{\sigma}\,\vec{u}_g \cdot \nabla\left(\frac{\partial \zeta_g}{\partial p}\right)}$$

The two underlined terms are **exactly equal in magnitude and opposite in sign**! Physically, these terms describe the geostrophic wind simply *translating* a pattern that is already in thermal wind balance, which, as we saw in the previous notes (where warm advection and differential vorticity advection canceled out for an atmosphere with no shear and in thermal wind balance), requires no vertical motion at all. As a result, there is often substantial **cancellation** between S_1 and S_3, so that the sign of ω depends on a small residual that is hard to determine by eye.

The **Q-vector** formulation rewrites the right-hand side of the omega equation so that this cancellation is built in from the start, making the sign of the right-hand-side much clearer.

How Does Thermal Wind Balance Change in Time?

Armed with our knowledge that vertical motion exists to restore thermal wind balance, let us consider the thermal wind balance equations, which we recall in pressure coordinates:

$$\text{Thermal wind balance:} \qquad f_0 \frac{\partial v_g}{\partial p} = \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial p}\right) = -\frac{R}{p}\frac{\partial T}{\partial x} \tag{2}$$

$$f_0 \frac{\partial u_g}{\partial p} = -\frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial p}\right) = \frac{R}{p}\frac{\partial T}{\partial y} \tag{3}$$

As we learned earlier, these equations tell us nothing about the temporal evolution of the atmosphere, they simply tell us a balance between horizontal temperature gradients and vertical wind shear that must exist when the Rossby number is small.

Again, as we learned earlier, the quasi-geostrophic equations tell us the temporal evolution of the system. Let us recall the quasi-geostrophic thermodynamic and momentum equations in pressure coordinates:

$$\text{QG thermodynamic equation:} \qquad \frac{D_g T}{Dt} - S_p\,\omega = 0 \tag{4}$$

$$\text{QG momentum equations:} \qquad \frac{D_g \vec{u}_g}{Dt} = -f_0\,\hat{k} \times \vec{u}_a \tag{5}$$

where $D_g/Dt = \partial/\partial t + u_g\,\partial/\partial x + v_g\,\partial/\partial y$, and $\sigma = RS_p/p$. Recall that the quasi-geostrophic momentum equations were derived from our asymptotic expansion in Rossby number, and in component form, they read

$$\frac{D_g u_g}{Dt} = f_0 v_a, \qquad \frac{D_g v_g}{Dt} = -f_0 u_a$$

Now, because we interpreted vertical motion as necessary to **restore the atmosphere to thermal wind balance**, we would like to know how the two sides of the thermal wind balance equation evolve in time, i.e.

$$\frac{D_g}{Dt}\left(f_0 \frac{\partial v_g}{\partial p}\right) \stackrel{?}{=} \frac{D_g}{Dt}\left(-\frac{R}{p}\frac{\partial T}{\partial x}\right) \tag{6}$$

where the same can be done for thermal wind balance in the other direction. In order to write an equation for the time evolution of thermal wind balance, we will need to take $\partial/\partial x$ of Eq. 4, take $\partial/\partial p$ of Eq. 5, and compare the two.

The Time Evolution of the Zonal Temperature Gradient

Taking $\partial/\partial x$ of the thermodynamic equation, and recalling that S_p depends only on p,

$$\frac{\partial}{\partial x}\left(\frac{\partial T}{\partial t} + u_g \frac{\partial T}{\partial x} + v_g \frac{\partial T}{\partial y}\right) = S_p \frac{\partial \omega}{\partial x}$$

Applying the product rule,

$$\underbrace{\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial x}\right) + u_g \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right) + v_g \frac{\partial}{\partial y}\left(\frac{\partial T}{\partial x}\right)}_{\frac{D_g}{Dt}\left(\frac{\partial T}{\partial x}\right)} + \frac{\partial u_g}{\partial x}\frac{\partial T}{\partial x} + \frac{\partial v_g}{\partial x}\frac{\partial T}{\partial y} = S_p \frac{\partial \omega}{\partial x}$$

Multiplying by $-R/p$ (which can be brought inside D_g/Dt, since p is constant along the geostrophic motion), and using $\sigma = RS_p/p$,

$$\boxed{\frac{D_g}{Dt}\left(-\frac{R}{p}\frac{\partial T}{\partial x}\right) = -Q_1 - \sigma\frac{\partial \omega}{\partial x}, \qquad \text{where} \qquad Q_1 \equiv -\frac{R}{p}\left(\frac{\partial u_g}{\partial x}\frac{\partial T}{\partial x} + \frac{\partial v_g}{\partial x}\frac{\partial T}{\partial y}\right)} \tag{7}$$

Notice that the left-hand-side of the Eq. 7 is exactly the right-hand-side of Eq. 6. Furthermore, notice that Q_1 contains only **geostrophic** quantities. It represents the tendency of the geostrophic flow to change the zonal temperature gradient, following the geostrophic motion (stay tuned!).

The Time Evolution of the Meridional Wind Shear

Next, we take $\partial/\partial p$ of the meridional quasi-geostrophic momentum equation, $D_g v_g/Dt = -f_0 u_a$:

$$\frac{D_g}{Dt}\left(\frac{\partial v_g}{\partial p}\right) + \frac{\partial u_g}{\partial p}\frac{\partial v_g}{\partial x} + \frac{\partial v_g}{\partial p}\frac{\partial v_g}{\partial y} = -f_0\frac{\partial u_a}{\partial p}$$

Multiplying by f_0,

$$\frac{D_g}{Dt}\left(f_0\frac{\partial v_g}{\partial p}\right) = -\underbrace{f_0\frac{\partial u_g}{\partial p}}_{①}\frac{\partial v_g}{\partial x} - \underbrace{f_0\frac{\partial v_g}{\partial p}}_{②}\underbrace{\frac{\partial v_g}{\partial y}}_{③} - f_0^2\frac{\partial u_a}{\partial p}$$

We will have to do more work to rewrite this into reasonable form.Notice that both ① and ② contain terms from the thermal wind balance equations (Eqs. 2 and 3). Furthermore notice that ③ can be rewritten using the fact that the geostrophic wind is non-divergent, or $\partial v_g/\partial y = -\partial u_g/\partial x$.

Let's use thermal wind balance and the non-divergence of $\vec{u}_g$ to substitute ①, ②, and ③:

$$\frac{D_g}{Dt}\left(f_0\frac{\partial v_g}{\partial p}\right) = -\frac{R}{p}\left(\frac{\partial T}{\partial y}\frac{\partial v_g}{\partial x} + \frac{\partial T}{\partial x}\frac{\partial u_g}{\partial x}\right) - f_0^2\frac{\partial u_a}{\partial p}$$

Notice that the first term on the right-hand side is exactly Q_1! Thus,

$$\boxed{\frac{D_g}{Dt}\left(f_0\frac{\partial v_g}{\partial p}\right) = Q_1 - f_0^2\frac{\partial u_a}{\partial p}} \tag{8}$$

We now have time-tendency equations for both sides of thermal wind balance:

$$\frac{D_g}{Dt}\left(-\frac{R}{p}\frac{\partial T}{\partial x}\right) = -Q_1 - \sigma\frac{\partial \omega}{\partial x} \qquad \text{(zonal temperature gradient)}$$

$$\frac{D_g}{Dt}\left(f_0\frac{\partial v_g}{\partial p}\right) = Q_1 - f_0^2\frac{\partial u_a}{\partial p} \qquad \text{(meridional wind shear)}$$

For thermal wind balance to be maintained, the two left-hand sides must be exactly equal. But the geostrophic term, Q_1, appears with **opposite signs** in the two equations! Without the ageostrophic terms (ω, u_a), the geostrophic flow would push the temperature gradient in one direction while simultaneously pushing the vertical wind shear in the other direction, disrupting thermal wind balance. This is a very powerful result:

Thermal wind balance is **continuously destroyed by the very circulation that is in balance**. An ageostrophic circulation (u_a, v_a, ω) must therefore develop to restore it.

What must the ageostrophic circulation be? Setting the two left-hand sides equal (i.e., subtracting Eq. 7 from Eq. 8),

$$0 = 2Q_1 - f_0^2 \frac{\partial u_a}{\partial p} + \sigma \frac{\partial \omega}{\partial x} \qquad \Longrightarrow \qquad \boxed{2Q_1 = f_0^2 \frac{\partial u_a}{\partial p} - \sigma \frac{\partial \omega}{\partial x}} \tag{9}$$

This equation dictates what the ageostrophic circulation (u_a, ω) needs to be in order for the atmosphere to re-attain thermal wind balance after the disruption from geostrophic advection!

The Time Evolution of the Meridional Temperature Gradient

To complete our system, we need to perform the same procedure, except for the other dimension in which thermal wind balance holds. Repeating the same steps for the meridional temperature gradient by taking $\partial/\partial y$ of the thermodynamic equation and multiplying by R/p, we obtain

$$\boxed{\frac{D_g}{Dt}\left(\frac{R}{p}\frac{\partial T}{\partial y}\right) = Q_2 + \sigma \frac{\partial \omega}{\partial y}, \qquad \text{where} \qquad Q_2 \equiv -\frac{R}{p}\left(\frac{\partial u_g}{\partial y}\frac{\partial T}{\partial x} + \frac{\partial v_g}{\partial y}\frac{\partial T}{\partial y}\right)} \tag{10}$$

The Time Evolution of the Zonal Wind Shear

As in the derivation of the time evolution of the meridional wind shear equation, we can take $\partial/\partial p$ of the zonal momentum equation, $D_g u_g/Dt = f_0 v_a$, multiplying by f_0, and using thermal wind balance and non-divergence as before, to obtain

$$\boxed{\frac{D_g}{Dt}\left(f_0 \frac{\partial u_g}{\partial p}\right) = -Q_2 + f_0^2 \frac{\partial v_a}{\partial p}} \tag{11}$$

Once again, we see that the geostrophic term appears on the right-hand-sides of Eqs. 10 and 11 with opposite signs ($+Q_2$ and $-Q_2$). Our interpretation is again the same - the geostrophic wind acts to destroy thermal wind balance.

Thermal wind balance requires $f_0\, \partial u_g/\partial p = \frac{R}{p}\, \partial T/\partial y$, so the two left-hand sides of Eqs. 10 and 11 must be equal if thermal wind balance were to be maintained at all times. Requiring this balance to hold yields,

$$Q_2 + \sigma \frac{\partial \omega}{\partial y} = -Q_2 + f_0^2 \frac{\partial v_a}{\partial p} \qquad \Longrightarrow \qquad \boxed{2Q_2 = f_0^2 \frac{\partial v_a}{\partial p} - \sigma \frac{\partial \omega}{\partial y}} \tag{12}$$

Notice the similarities between Eqs. 9 and 12. They are so similar, in fact, that we can combine them into a singular equation!

The Q-Vector Form of the Omega Equation

We can combine Eqs. 9 and 12 into a single vector equation,

$$\boxed{2\vec{Q} = f_0^2 \frac{\partial \vec{u}_a}{\partial p} - \sigma \nabla \omega, \qquad \text{where} \qquad \vec{Q} = (Q_1, Q_2) = -\frac{R}{p}\left(\frac{\partial \vec{u}_g}{\partial x} \cdot \nabla T,\ \frac{\partial \vec{u}_g}{\partial y} \cdot \nabla T\right)} \tag{13}$$

How do we use Eq. 13? Notice that $\vec{Q}$ is determined entirely by the geostrophic flow, such that the left-hand-side is determined entirely by the geostrophic flow. Furthermore, notice that the right-hand-side is determined entirely by the ageostrophic flow. Thus, given the geostrophic flow (the

left-hand-side), we can use these equations to determine the ageostrophic circulation (u_a, v_a, ω)! Aha! – we've been searching for this equation since time we derived the quasi-geostrophic vorticity equation. We can actually further simplify this by taking its horizontal divergence:

$$2\nabla \cdot \vec{Q} = f_0^2 \frac{\partial}{\partial p} \left(\nabla \cdot \vec{u}_a \right) - \sigma \nabla^2 \omega$$

Recall from mass conservation in pressure coordinates that $\nabla \cdot \vec{u}_a + \partial\omega/\partial p = 0$, since the geostrophic wind is non-divergent. Substituting,

$$2\nabla \cdot \vec{Q} = -f_0^2 \frac{\partial^2 \omega}{\partial p^2} - \sigma \nabla^2 \omega$$

and dividing by $-\sigma$, we obtain the **Q-vector form of the quasi-geostrophic omega equation**:

$$\boxed{\left(\nabla^2 + \frac{f_0^2}{\sigma} \frac{\partial^2}{\partial p^2} \right) \omega = -\frac{2}{\sigma} \nabla \cdot \vec{Q}} \tag{14}$$

Notice that the left-hand side is **identical** to the left-hand-side of the (traditional) omega equation we derived a few lectures ago. However, the right-hand-side has been completely rewritten. Still, we may interpret ω as the vertical velocity that exists in order to restore thermal wind balance. As noted earlier, this can be seen quantitatively by comparing Eq. 7 to Eq. 8, and Eq. 10 to 11. In summary,

$$\boxed{\begin{array}{c} \textbf{Q-vector form: } \left(\nabla^2 + \frac{f_0^2}{\sigma} \frac{\partial^2}{\partial p^2} \right) \omega = -\frac{2}{\sigma} \nabla \cdot \vec{Q} \\ \textbf{Traditional form: } \left(\nabla^2 + \frac{f_0^2}{\sigma} \frac{\partial^2}{\partial p^2} \right) \omega = -\frac{f_0}{\sigma} \frac{\partial}{\partial p} \left(-\vec{u}_g \cdot \nabla \zeta_g \right) + \frac{f_0 \beta}{\sigma} \frac{\partial v_g}{\partial p} - \frac{1}{\sigma} \nabla^2 \left[-\vec{u}_g \cdot \nabla \left(-\frac{\partial \phi}{\partial p} \right) \right] \end{array}}$$

A few remarks on the Q-vector form:

- The Q-vector form tends to **obscure** the physical interpretation of ω, but it fixes the cancellation issue of the traditional form, since the canceling terms never appear.
- $-\frac{2}{\sigma} \nabla \cdot \vec{Q}$ is easily calculated from gridded model data, as it depends only on the geostrophic wind and the temperature on a *single* pressure level. In contrast, the quasi-geostrophic omega equation requires vertical derivatives to calculate differential vorticity advection.
- If we had kept β, an additional term $\frac{f_0 \beta}{\sigma} \partial v_g / \partial p$ would appear on the right-hand side, just as in the traditional quasi-geostrophic omega equation.

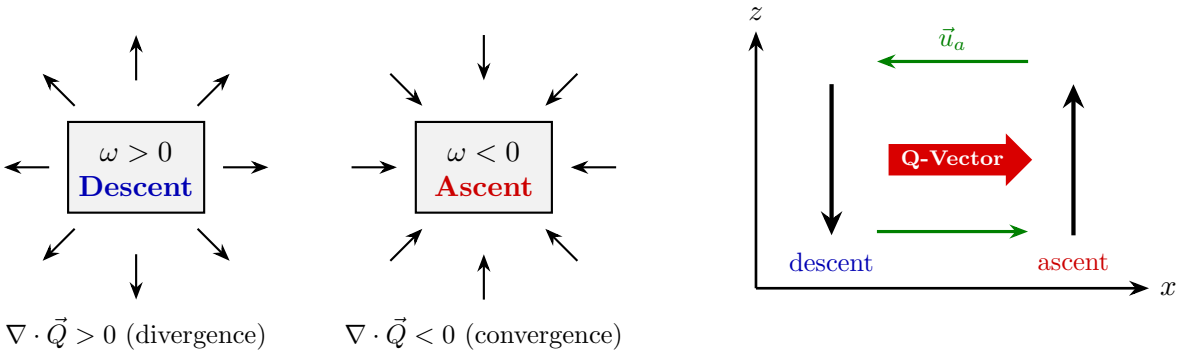
Interpreting Q-Vectors

Since the operator on the left-hand side of Eq. (14) is elliptic, ω is anticorrelated with the right-hand-side, $-\frac{2}{\sigma} \nabla \cdot \vec{Q}$, which means that

$$\omega \ \propto \ \nabla \cdot \vec{Q}$$

- Where the Q-vectors **converge** ($\nabla \cdot \vec{Q} < 0$), there is **ascent** ($\omega < 0$).
- Where the Q-vectors **diverge** ($\nabla \cdot \vec{Q} > 0$), there is **descent** ($\omega > 0$).

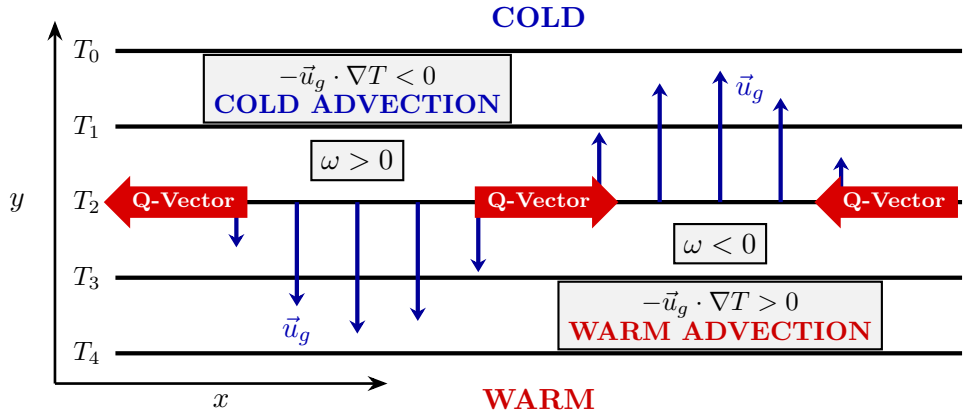
Q-vectors point away from regions of descent, and towards regions of ascent.



The vertical cross-section on the right shows the full ageostrophic circulation implied by $2\vec{Q} = f_0^2 \, \partial \vec{u}_a / \partial p - \sigma \nabla \omega$: the Q-vector points from descent ($\omega > 0$) toward ascent ($\omega < 0$). Next, we will consider some examples to better understand Q-vectors.

Example 1: Meridional Flow Across Zonal Isotherms

Consider zonally oriented isotherms (cold in the north), and a purely meridional geostrophic wind that varies only in x: southward on the left, and northward on the right, as indicated below:



Then $u_g = 0$, $\partial T / \partial x = 0$, and $\partial v_g / \partial y = 0$, so most terms in $\vec{Q}$ vanish:

$$\vec{Q} = -\frac{R}{p}\left[\underbrace{\frac{\partial u_g}{\partial x}}_{0}\underbrace{\frac{\partial T}{\partial x}}_{0} + \frac{\partial v_g}{\partial x}\frac{\partial T}{\partial y}, \; \underbrace{\frac{\partial u_g}{\partial y}}_{0}\underbrace{\frac{\partial T}{\partial x}}_{0} + \underbrace{\frac{\partial v_g}{\partial y}}_{0}\frac{\partial T}{\partial y}\right] = \left[-\frac{R}{p}\frac{\partial v_g}{\partial x}\underbrace{\frac{\partial T}{\partial y}}_{<0}, \; 0\right] \propto \left[\frac{\partial v_g}{\partial x}, \; 0\right]$$

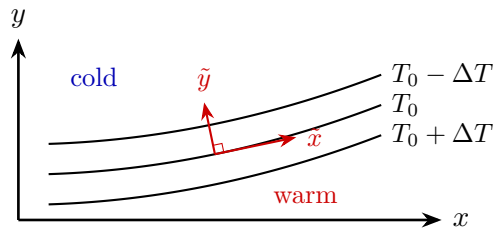
The final step makes use of the fact that the meridional temperature gradient is negative, such that the Q-vector has no y-component, and its x-component points in the direction of $\partial v_g / \partial x$. Therefore, the Q-vector points west where v_g decreases with x, and east where v_g increases with x, as indicated in the diagram.

In our set-up, the Q-vectors diverge in the region of southward flow (cold advection), giving **descent**, and converge in the region of northward flow (warm advection), giving **ascent**. (Is this consistent with what we would have deduced from the traditional omega equation?)

A Graphical Shortcut

In practice, it is not easy to compute $\vec{Q}$ just by looking at a weather map, which usually has more complicated flows. Fortunately, there is a shortcut. Let us rotate our horizontal coordinates so:

- $\tilde{x}$ points **along** the isotherms (**with cold air to the left**), and
- $\tilde{y}$ points **across** the isotherms, toward cold air (i.e., parallel to $-\nabla T$).



The $\tilde{x}$-direction is often called the **thermal wind coordinate**. In these coordinates, $\partial T/\partial \tilde{x} = 0$ and $\partial T/\partial \tilde{y} = -|\nabla T|$, so

$$\vec{Q} = -\frac{R}{p}\left[\frac{\partial \tilde{u}_g}{\partial \tilde{x}}\underbrace{\frac{\partial T}{\partial \tilde{x}}}_{0} + \frac{\partial \tilde{v}_g}{\partial \tilde{x}}\frac{\partial T}{\partial \tilde{y}},\ \frac{\partial \tilde{u}_g}{\partial \tilde{y}}\underbrace{\frac{\partial T}{\partial \tilde{x}}}_{0} + \frac{\partial \tilde{v}_g}{\partial \tilde{y}}\frac{\partial T}{\partial \tilde{y}}\right] = \frac{R}{p}\frac{\partial T}{\partial \tilde{y}}\left[-\frac{\partial \tilde{v}_g}{\partial \tilde{x}},\ \frac{\partial \tilde{u}_g}{\partial \tilde{x}}\right]$$

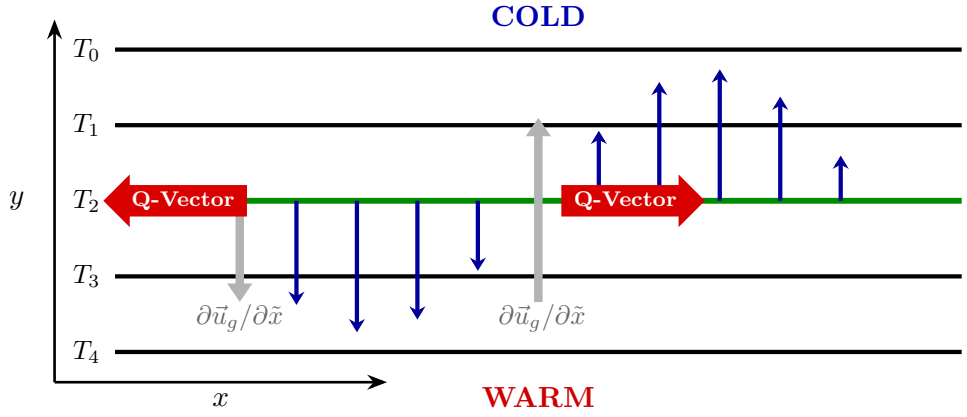
where we have used non-divergence, $\partial \tilde{v}_g/\partial \tilde{y} = -\partial \tilde{u}_g/\partial \tilde{x}$. Recognizing $[-a_2, a_1] = \hat{k} \times \vec{a}$,

$$\boxed{\vec{Q} = -\frac{R}{p}\,|\nabla T|\ \hat{k} \times \frac{\partial \vec{u}_g}{\partial \tilde{x}}} \tag{15}$$

Since $\hat{k}\times$ rotates a vector 90° counter-clockwise, the minus sign rotates it 90° *clockwise*, i.e., anti-cyclonically in the Northern Hemisphere. This gives a simple trick for determing the Q-vector:

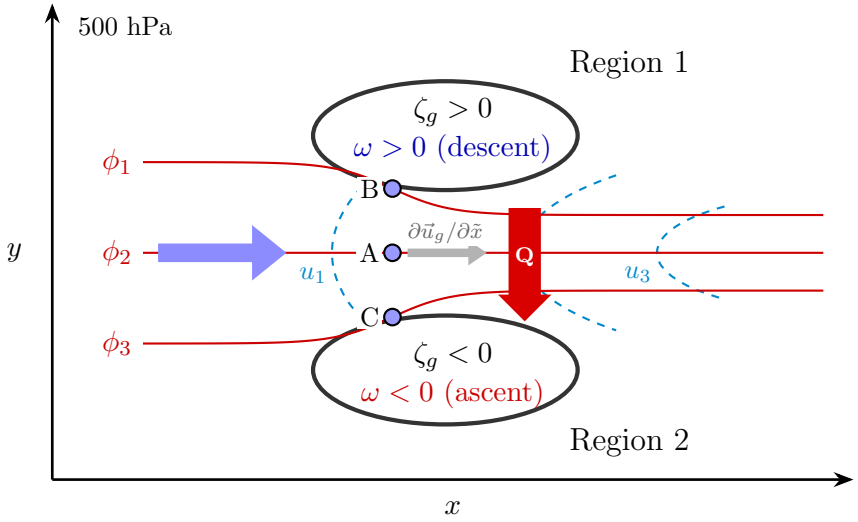
> Take the **vector change of** $\vec{u}_g$ along the isotherms, moving forwards with cold air on the left, and **rotate it anticyclonically by** 90°. The magnitude scales with $|\nabla T|$.

Applying this to Example 1, along the T_2 isotherm (moving east), the change in $\vec{u}_g$ points south on the left side (where the southward wind is strengthening) and north in the middle (where the wind turns from southward to northward). Rotating these anticyclonically by 90° gives Q-vectors pointing west and east, respectively, exactly as before. See the diagram below.



Example 2: Returning to the Jet Entrance Region

Let us revisit the jet entrance region from the quasi-geostrophic omega notes. As before, suppose the 1000-hPa geopotential is flat, so that the 500-hPa geopotential contours are also isotherms (cold to the north). Following the isotherm through Point A toward the east, the geostrophic wind *increases* as the flow enters the jet, so $\partial\vec{u}_g/\partial\tilde{x}$ points **east**. Rotating it anticyclonically by 90°, the Q-vector at A points **south**, from Region 1 toward Region 2:



We can confirm this mathematically. At Point A, the flow is purely zonal and symmetric about the jet axis, so $\partial v_g/\partial x = 0$, and the isotherms are zonal, so $\partial T/\partial x = 0$. The flow accelerates into the jet, so $\partial u_g/\partial x > 0$, and by non-divergence $\partial v_g/\partial y = -\partial u_g/\partial x < 0$. Thus,

$$\vec{Q} = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = -\frac{R}{p} \begin{bmatrix} \frac{\partial u_g}{\partial x}\underbrace{\frac{\partial T}{\partial x}}_{0} + \underbrace{\frac{\partial v_g}{\partial x}}_{0}\frac{\partial T}{\partial y} \\ \frac{\partial u_g}{\partial y}\underbrace{\frac{\partial T}{\partial x}}_{0} + \underbrace{\frac{\partial v_g}{\partial y}}_{<0}\underbrace{\frac{\partial T}{\partial y}}_{<0} \end{bmatrix} \qquad \Longrightarrow \qquad Q_1 = 0, \quad Q_2 < 0$$

The Q-vector points south, so it diverges from Region 1 (**descent**, $\omega > 0$) and converges into Region 2 (**ascent**, $\omega < 0$). Notice how we obtain the exact same vertical motion from Q-vectors as from the traditional QG omega equation!

The same reasoning applies at the **jet exit**, where the flow decelerates as it leaves the jet. There, $\partial\vec{u}_g/\partial\tilde{x}$ points **west**, so the Q-vector points **north**: ascent on the poleward side (the left exit) and descent on the equatorward side (the right exit). The ascent in the left exit is accompanied by divergence aloft and convergence (vortex stretching) below, which is why the left exit region of a jet streak is a favored location for surface cyclone development. Next, let's take a closer look at the ageostrophic circulation in the jet cross-section.

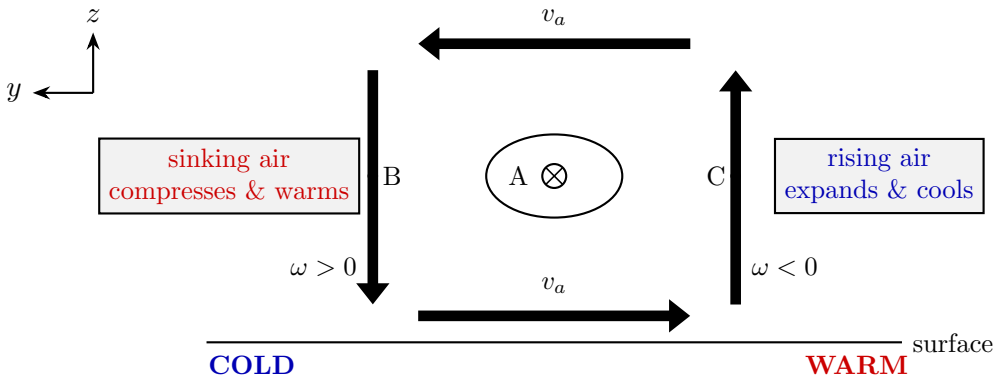
The Ageostrophic Circulation Across the Jet

Since $Q_1 = 0$ and $Q_2 < 0$ at Point A, let us look at the two equations for the meridional temperature gradient and the vertical wind shear (Eqs. 10 and 11). It is convenient to multiply Eq. (11) by -1,

since $-\partial u_g/\partial p > 0$ when the westerlies increase with height:

$$\frac{D_g}{Dt}\left(\frac{R}{p}\frac{\partial T}{\partial y}\right) = Q_2 + \sigma\frac{\partial \omega}{\partial y}, \qquad \frac{D_g}{Dt}\left(-f_0\frac{\partial u_g}{\partial p}\right) = Q_2 - f_0^2\frac{\partial v_a}{\partial p}$$

Thermal wind balance requires $-f_0\,\partial u_g/\partial p = -\frac{R}{p}\,\partial T/\partial y$: the westerly shear (positive) must match the poleward temperature decrease (also positive). The corresponding cross-section across the jet (with the jet flowing into the page at A) is:



The meridional temperature gradient. Since $Q_2 < 0$, the geostrophic flow is **further decreasing** $\partial T/\partial y$, which is already negative (cold to the north). In other words, following the flow into the confluent jet entrance, the isotherms are squeezed together, and the temperature gradient strengthens. To oppose this, we need $\sigma\,\partial\omega/\partial y > 0$, i.e., $\partial\omega/\partial y > 0$, or descent ($\omega > 0$) on the cold (poleward) side at B and ascent ($\omega < 0$) on the warm (equatorward) side at C. The adiabatic warming of the sinking air and adiabatic cooling of the rising air weakens the meridional temperature gradient, opposing the effects of the geostrophic wind (Q_2).

The vertical wind shear. Since $Q_2 < 0$, the geostrophic deformation simultaneously acts to **decrease** the westerly vertical wind shear, $-f_0\,\partial u_g/\partial p$. To restore thermal wind balance, we need $-f_0^2\,\partial v_a/\partial p > 0$, i.e., $\partial v_a/\partial p < 0$, or an increasingly poleward ageostrophic wind with increasing height. For example, a poleward ageostrophic wind aloft and an equatorward ageostrophic wind near the surface. Recall that

$$\frac{D_g u_g}{Dt} = f_0 v_a$$

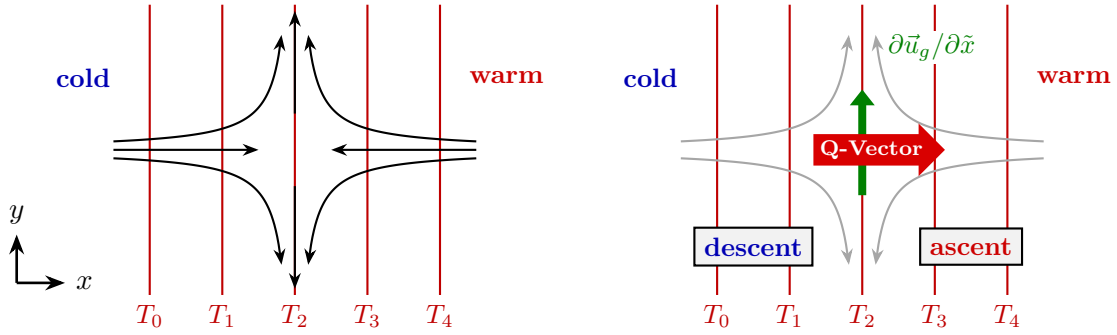
so the poleward ageostrophic wind aloft **accelerates** u_g at A (imposing an additional Coriolis force in the u_g direction), while the equatorward ageostrophic wind near the surface decelerates u_g (imposing an additional Coriolis force against the u_g direction). Thus, $\partial v_a/\partial p < 0$ **increases** the westerly shear over time, again opposing the effects of the geostrophic wind (Q_2).

Notice that both the temperature gradient and the wind shear are being pushed in the wrong direction by the geostrophic flow, which makes the shear too weak for the temperature gradient. This is the same conclusion we reached in the previous notes, where we argued at a fixed point (rather than following the flow) that the shear decreases while the temperature gradient stays fixed. The cross-frontal ageostrophic circulation, which can be diagnosed from the Q-vector equation, is the thermally direct circulation that restores thermal wind balance.

Both the quasi-geostrophic omega equation and the Q-vector framework show how the **cross-frontal ageostrophic circulation** restores thermal wind balance, which is continuously destroyed by the geostrophic wind!

Example 3: A Deformation Field

Consider meridionally oriented isotherms (cold to the west), and a geostrophic **deformation** field that brings air together along the x-axis and spreads it apart along the y-axis:



With cold air to the west, $\tilde{y}$ points west and $\tilde{x}$ points **north**, along the isotherms. Moving north through the center, the geostrophic wind changes from southward to northward, so $\partial\vec{u}_g/\partial\tilde{x}$ points **north**. Rotating anticyclonically by 90°, the Q-vector points **east**, from the cold air toward the warm air. Since the deformation (and thus $\vec{Q}$) is strongest near the center, the Q-vectors diverge on the cold side (**descent**) and converge on the warm side (**ascent**), as shown on the right.

This is a thermally direct circulation (warm air rises, cold air sinks). Notice also that this deformation field squeezes the isotherms together, strengthening the temperature gradient. Such a setup is called **frontogenetic**. Stay tuned for frontogenesis!

Major Themes of This Unit

- The geostrophic flow **continually** acts to destroy thermal wind balance. This is because the vorticity and temperature advection by the geostrophic circulation disrupts the balance between the vertical wind shear and the horizontal temperature gradient.
- The quasi-geostrophic omega equation (or, equivalently, the Q-vector form) tells you the **ageostrophic circulation** (u_a, v_a, ω) that is necessary to **restore thermal wind balance**. In other words, the ageostrophic circulation exists to undo the destruction of thermal wind balance done by the geostrophic circulation!
- Q-vectors point towards ascent and away from descent: $\omega \propto \nabla \cdot \vec{Q}$. Graphically, $\vec{Q}$ is the change in $\vec{u}_g$ along the isotherms, rotated anticyclonically by 90°.

The Q-Vector Form of the QG Omega Equation

$$\left(\nabla^2 + \frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega = -\frac{2}{\sigma}\nabla\cdot\vec{Q} \qquad \omega \propto \nabla\cdot\vec{Q}$$

$$\vec{Q} = -\frac{R}{p}\left(\frac{\partial\vec{u}_g}{\partial x}\cdot\nabla T,\ \frac{\partial\vec{u}_g}{\partial y}\cdot\nabla T\right) = -\frac{R}{p}\left|\nabla T\right|\hat{k}\times\frac{\partial\vec{u}_g}{\partial\tilde{x}}$$

$$2\vec{Q} = f_0^2\frac{\partial\vec{u}_a}{\partial p} - \sigma\nabla\omega$$

Q-vectors point toward ascent and away from descent. Graphically, take the change in $\vec{u}_g$ along the isotherms (cold air on the left) and rotate it anticyclonically by 90°.

Frontogenesis and Semi-Geostrophy

Chapter 10

Last Modified: November 15, 2026

Where We Left Off

In the last unit, we found that the geostrophic flow **continually** acts to destroy thermal wind balance, and that the ageostrophic circulation (u_a, v_a, ω) develops to restore it. In particular, using the Q-vector framework (in pressure coordinates, on an f-plane), we derived equations for how the geostrophic flow changes the horizontal temperature gradient. For example, for the meridional temperature gradient,

$$\boxed{\frac{D_g}{Dt}\left(\frac{R}{p}\frac{\partial T}{\partial y}\right) = Q_2 + \sigma\frac{\partial \omega}{\partial y}, \qquad \text{where} \qquad Q_2 = -\frac{R}{p}\left(\frac{\partial u_g}{\partial y}\frac{\partial T}{\partial x} + \frac{\partial v_g}{\partial y}\frac{\partial T}{\partial y}\right)} \tag{1}$$

In these notes, we will use this equation to understand how **fronts** form. We will find that quasi-geostrophy gets us part of the way there, but that it breaks down in the cross-frontal direction.

Introduction to Frontogenesis

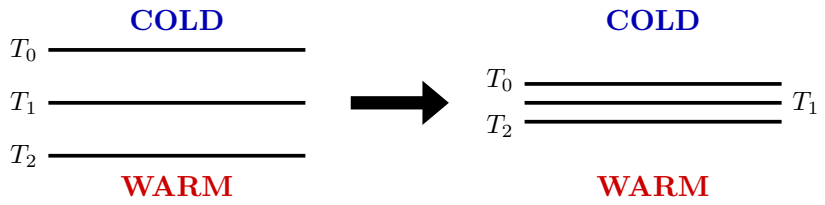
Fronts are among the most familiar features of mid-latitude weather. On a weather map of a developing synoptic-scale storm, the precipitation, the 500-hPa vorticity, and the lower-tropospheric temperature are all organized along narrow, elongated bands. A vertical cross-section taken *across* one of these bands reveals that the temperature can change by as much as $\sim$15 °C over only $\sim$150 km, and that this zone of strong temperature gradient is **slanted**: it tilts with height toward the cold air. At the surface, the passage of such a front is often marked by a dramatic shelf cloud and a sharp change in wind and temperature.

Frontogenesis is the formation and intensification of a weather front, i.e., a region of strong gradients of an atmospheric quantity (such as temperature) over a short distance. In these notes, we will try to answer the following questions:

- What are the physical processes that lead to frontogenesis?
- What determines the **time scale** at which fronts form in synoptic-scale storms?
- What determines the **spatial scale** of these fronts?
- What determines the **shape** (e.g., the slant) of these fronts?

Quantifying Frontogenesis

Consider the intensification of a temperature front in the y-direction, i.e., the isotherms are oriented along x, and they become more tightly packed over time:



We can use the Q-vector equation for the meridional temperature gradient (Eq. 1) to understand its time evolution. Dividing by R/p (which is constant following the geostrophic motion, since it is horizontal), and recalling that $\sigma = RS_p/p$,

$$\frac{D_g}{Dt}\left(\frac{\partial T}{\partial y}\right) = \frac{p}{R}Q_2 + S_p\frac{\partial \omega}{\partial y}, \qquad \text{where} \qquad \frac{p}{R}Q_2 = -\frac{\partial u_g}{\partial y}\frac{\partial T}{\partial x} - \frac{\partial v_g}{\partial y}\frac{\partial T}{\partial y}$$

Since the geostrophic wind is non-divergent, $\partial v_g/\partial y = -\partial u_g/\partial x$, so that

$$\frac{p}{R}Q_2 = -\frac{\partial u_g}{\partial y}\frac{\partial T}{\partial x} + \frac{\partial u_g}{\partial x}\frac{\partial T}{\partial y}$$

Combining, we obtain:

$$\boxed{\frac{D_g}{Dt}\left(\frac{\partial T}{\partial y}\right) = -\left[\underbrace{\frac{\partial u_g}{\partial y}\frac{\partial T}{\partial x}}_{(1)} - \underbrace{\frac{\partial u_g}{\partial x}\frac{\partial T}{\partial y}}_{(2)}\right] + \underbrace{S_p\frac{\partial \omega}{\partial y}}_{(3)}} \tag{2}$$

The three terms represent:

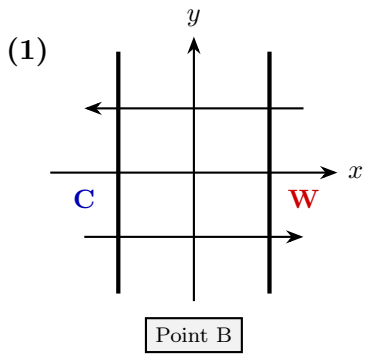
1. **Horizontal shearing deformation** by the **geostrophic** circulation;
2. **Horizontal stretching deformation** by the **geostrophic** circulation;
3. **Differential adiabatic warming/cooling** by the **ageostrophic** circulation.

Similarly, taking $\partial/\partial x$ instead, the zonal temperature gradient evolves as

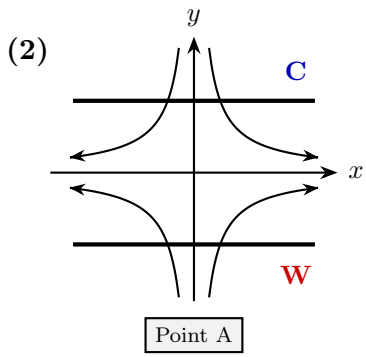
$$\boxed{\frac{D_g}{Dt}\left(\frac{\partial T}{\partial x}\right) = -\left[\underbrace{\frac{\partial v_g}{\partial x}\frac{\partial T}{\partial y}}_{(1)} - \underbrace{\frac{\partial v_g}{\partial y}\frac{\partial T}{\partial x}}_{(2)}\right] + \underbrace{S_p\frac{\partial \omega}{\partial x}}_{(3)}} \tag{3}$$

Horizontal Deformation

The two geostrophic terms correspond to two different flow configurations:



shearing deformation



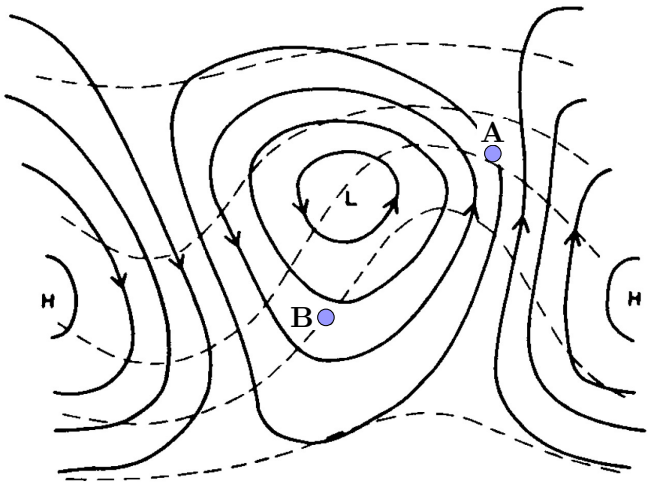
stretching deformation

(1) rotates (deforms) a zonal temperature gradient into the meridional direction.

(2) advects the temperature field so that the isotherms concentrate along the dilation axis (here, the x-axis).

Horizontal Deformation in Synoptic Storms

In a developing synoptic-scale storm, the large-scale geostrophic circulation contains both types of deformation. In the schematic below (from Holton), the solid lines are streamlines of the geostrophic flow around a surface low, and the dashed lines are isotherms:



Source: Holton, *An Introduction to Dynamic Meteorology*, p. 271.

Point A: stretching deformation. Ahead of (northeast of) the low, the flow converges toward the isotherms along y and spreads out along x, much like configuration (2). Here, $\partial T/\partial y < 0$ (cold to the north) and $\partial u_g/\partial x > 0$, so the stretching term in Eq. (2) gives

$$\frac{D_g}{Dt}\left(\frac{\partial T}{\partial y}\right) \approx \underbrace{\frac{\partial u_g}{\partial x}}_{>0}\underbrace{\frac{\partial T}{\partial y}}_{<0} < 0$$

The meridional temperature gradient intensifies, and a classical **warm front** forms! Note that Point A is a region of warm air advection.

Point B: shearing deformation. South of the low, the northerly flow to the west and the southerly flow to the east shear the temperature field, much like configuration (1). Here, $\partial T/\partial y < 0$ and $\partial v_g/\partial x > 0$, so the shearing term in Eq. (3) gives

$$\frac{D_g}{Dt}\left(\frac{\partial T}{\partial x}\right) \approx -\underbrace{\frac{\partial v_g}{\partial x}}_{>0}\underbrace{\frac{\partial T}{\partial y}}_{<0} > 0$$

The zonal temperature gradient intensifies with cold air to the west, and a classical **cold front** forms! Note that Point B is a region of cold air advection.

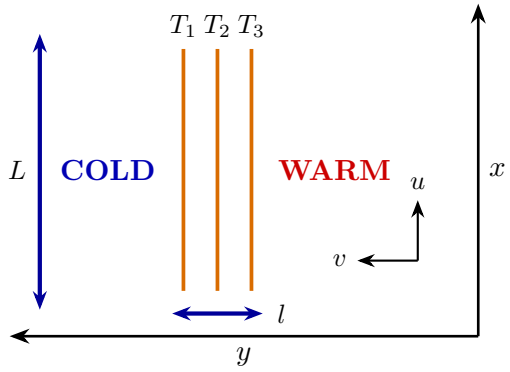
Horizontal deformation by the **geostrophic circulation** tends to:
- form **warm fronts** ahead of the surface low pressure, and
- form **cold fronts** southward of and behind the surface low pressure.

This **frontogenesis** drives the system away from thermal wind balance!

However, there are some issues with applying quasi-geostrophy in the cross-frontal region...

Scale Analysis of a Front

Consider an atmospheric front with the temperature gradient in the y-direction (so the front is oriented along x), with cold air to the north:



Typical scales are:

- u: **along**-front (zonal) flow, $U \sim 10$ m/s;
- v: **across**-front (meridional) flow, $V \sim 1$ m/s;
- L: along-front horizontal scale, ≈ 1000 km;
- l: across-front horizontal scale, ≈ 100 km;
- f_0: Coriolis parameter, $\approx 10^{-4}$ s^{-1}.

The time scale is set by the time it takes the cross-front flow to cross the front, $t \sim l/V$ (a lower limit).

(1) The meridional momentum equation is

$$\frac{Dv}{Dt} + f_0 u = -\frac{\partial \phi}{\partial y}$$

Comparing the acceleration to the Coriolis force acting on the along-front wind,

$$\frac{Dv/Dt}{f_0 u} \sim \frac{V^2}{l}\frac{1}{f_0 U} = \frac{V^2}{U^2}\,\mathrm{Ro} \approx 0.01$$

The along-front wind u is very much in geostrophic (and thus thermal wind) balance!

(2) The zonal momentum equation is

$$\frac{Du}{Dt} - f_0 v = -\frac{\partial \phi}{\partial x}$$

Comparing the acceleration to the Coriolis force acting on the cross-front wind,

$$\frac{Du/Dt}{f_0 v} \sim \frac{UV}{l}\frac{1}{f_0 V} = \frac{U}{f_0 l} \approx 1$$

The acceleration is the **same order** as the Coriolis term, so the cross-front wind v is **not** close to geostrophic balance.

Quasi-geostrophy fails for the across-front (v) wind!

Going Beyond Quasi-Geostrophy

Our scale analysis shows that QG fails in the cross-frontal direction. While the along-front wind is in near-geostrophic balance, the cross-front wind is not even quasi-geostrophic. In particular,

- $u = u_g$ is very accurate, but $v = v_g + v_a$, where v_a is the **same order of magnitude** as v_g!
- Consequently, the **advection by the ageostrophic wind cannot be neglected** in the cross-frontal direction.

In other words, the geostrophic material derivative D_g/Dt needs to become the full material derivative D/Dt:

$$\begin{aligned}\frac{D}{Dt} &= \frac{\partial}{\partial t} + (u_g + \cancel{u_a})\frac{\partial}{\partial x} + (v_g + v_a)\frac{\partial}{\partial y} + \omega\frac{\partial}{\partial p} \\ &= \underbrace{\frac{\partial}{\partial t} + u_g\frac{\partial}{\partial x} + v_g\frac{\partial}{\partial y}}_{D_g/Dt} + v_a\frac{\partial}{\partial y} + \omega\frac{\partial}{\partial p}\end{aligned}$$

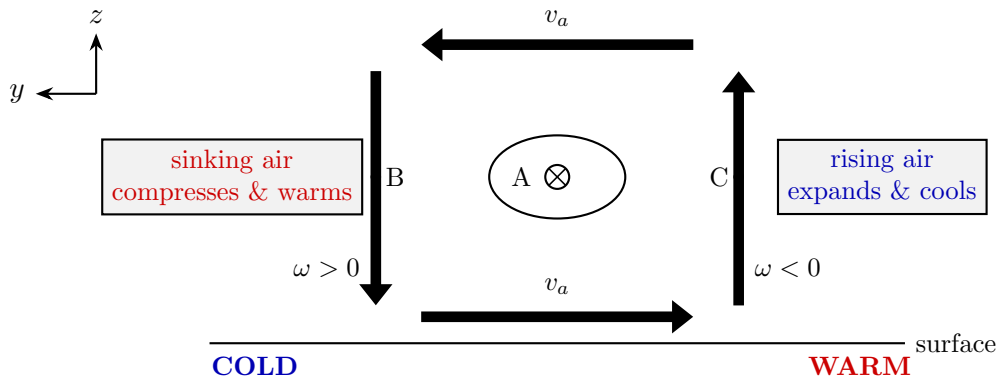
where we have neglected the along-front ageostrophic wind, u_a, since the along-front flow is in geostrophic balance. Including these extra advective terms is called **semi-geostrophic** balance.

Recap: The Q-Vector Equations

Recall from the Q-vector notes the two equations for the meridional temperature gradient and the vertical wind shear across a front oriented along x:

$$\frac{D_g}{Dt}\left(\frac{\partial T}{\partial y}\right) = \frac{p}{R}Q_2 + S_p\frac{\partial \omega}{\partial y}, \qquad \frac{D_g}{Dt}\left(-f_0\frac{\partial u_g}{\partial p}\right) = Q_2 - f_0^2\frac{\partial v_a}{\partial p}$$

Remember that Q_2 represents the deformation by the geostrophic flow! Consider $Q_2 < 0$, where horizontal deformation by the geostrophic flow **further decreases** $\partial T/\partial y$ (strengthening the front) while decreasing the westerly vertical wind shear. We found (through the QG omega equation and Q-vectors) that $\partial\omega/\partial y > 0$ and $\partial v_a/\partial p < 0$, which **restores thermal wind balance**:



In three dimensions, this cross-frontal ageostrophic circulation sits in the y–z plane, perpendicular to the along-front geostrophic flow u_g. However, in quasi-gesotrophy, the ageostrophic circulation does not advect temperature or momentum. Only the geostrophic wind appears in D_g/Dt. Our scale analysis shows that, near a front, this is not a good approximation.

The Semi-Geostrophic Equations

The semi-geostrophic equations allow the ageostrophic circulation to advect temperature and momentum. For a front oriented along x, the semi-geostrophic (SG) equations assume:

- The **along-front** flow is in geostrophic and thermal wind balance: $u = u_g$, where $u_g = -\dfrac{1}{f_0}\dfrac{\partial \phi}{\partial y}$, and $f_0 \dfrac{\partial u_g}{\partial p} = \dfrac{R}{p}\dfrac{\partial T}{\partial y}$
- The **cross-front** flow includes the ageostrophic wind: $v = v_g + v_a$, where $v_g = \dfrac{1}{f_0}\dfrac{\partial \phi}{\partial x}$.
- Material derivative: $\dfrac{D}{Dt} = \frac{D_g}{Dt} + \left(v_a \dfrac{\partial}{\partial y} + \omega \dfrac{\partial}{\partial p} \right)$
- Momentum equation: $\dfrac{D}{Dt} u_g - f_0 v_a = 0$
- Thermodynamic equation: $\dfrac{D}{Dt} T - S_p \, \omega = 0$
- Continuity equation: $\dfrac{\partial v_a}{\partial y} + \dfrac{\partial \omega}{\partial p} = 0$

These are identical to the quasi-geostrophic equations, except that the advection by the cross-frontal ageostrophic circulation (in red) has been retained. As in QG, the vertical advection of the basic-state temperature profile is contained in $S_p \, \omega$, so the T inside D/Dt is the departure from that basic state (its horizontal gradients are the same as those of the full temperature).

The Sawyer–Eliassen Circulation in Semi-Geostrophy

In semi-geostrophy,

- advection includes the **full** wind, and
- the cross-frontal ageostrophic circulation **advects temperature and momentum**.

Let us repeat our derivation of the Sawyer–Eliassen equations, now with the full material derivative. Taking $\partial/\partial y$ of the SG thermodynamic equation, the extra advective terms contribute

$$\frac{\partial}{\partial y}\left(v_a \frac{\partial T}{\partial y} + \omega \frac{\partial T}{\partial p} \right) = \left(v_a \frac{\partial}{\partial y} + \omega \frac{\partial}{\partial p} \right) \frac{\partial T}{\partial y} + \frac{\partial v_a}{\partial y}\frac{\partial T}{\partial y} + \frac{\partial \omega}{\partial y}\frac{\partial T}{\partial p}$$

The first group of terms simply turns D_g/Dt into D/Dt, while the last two are new. Similarly, taking $\partial/\partial p$ of the SG momentum equation, the extra advective terms contribute two new terms, $\partial v_a/\partial p \, \partial u_g/\partial y + \partial \omega/\partial p \, \partial u_g/\partial p$. Multiplying by $-f_0$, moving them to the right-hand side, and using thermal wind balance ($f_0 \, \partial u_g/\partial p = (R/p) \, \partial T/\partial y$) in the last term, we can compare the two sets of equations:

$$\left.\begin{aligned} \frac{D_g}{Dt}\left(\frac{\partial T}{\partial y} \right) &= \frac{p}{R} Q_2 + S_p \frac{\partial \omega}{\partial y} \\ \frac{D_g}{Dt}\left(-f_0 \frac{\partial u_g}{\partial p} \right) &= Q_2 - f_0^2 \frac{\partial v_a}{\partial p} \end{aligned}\right\} \text{QG}$$

$$\left.\begin{aligned} \frac{D}{Dt}\left(\frac{\partial T}{\partial y}\right) &= \frac{p}{R}Q_2 + S_p\frac{\partial \omega}{\partial y} - \frac{\partial v_a}{\partial y}\frac{\partial T}{\partial y} - \frac{\partial \omega}{\partial y}\frac{\partial T}{\partial p} \\ \frac{D}{Dt}\left(-f_0\frac{\partial u_g}{\partial p}\right) &= Q_2 - f_0^2\frac{\partial v_a}{\partial p} + f_0\frac{\partial v_a}{\partial p}\frac{\partial u_g}{\partial y} + \frac{\partial \omega}{\partial p}\frac{R}{p}\frac{\partial T}{\partial y} \end{aligned}\right\} \text{SG}$$

Semi-Geostrophic Frontogenesis

Let us focus on how the meridional temperature gradient changes:

$$\frac{D}{Dt}\left(\frac{\partial T}{\partial y}\right) = \frac{p}{R}Q_2 + S_p\frac{\partial \omega}{\partial y} \underbrace{- \frac{\partial v_a}{\partial y}\frac{\partial T}{\partial y}}_{\text{key new term}} - \frac{\partial \omega}{\partial y}\frac{\partial T}{\partial p}$$

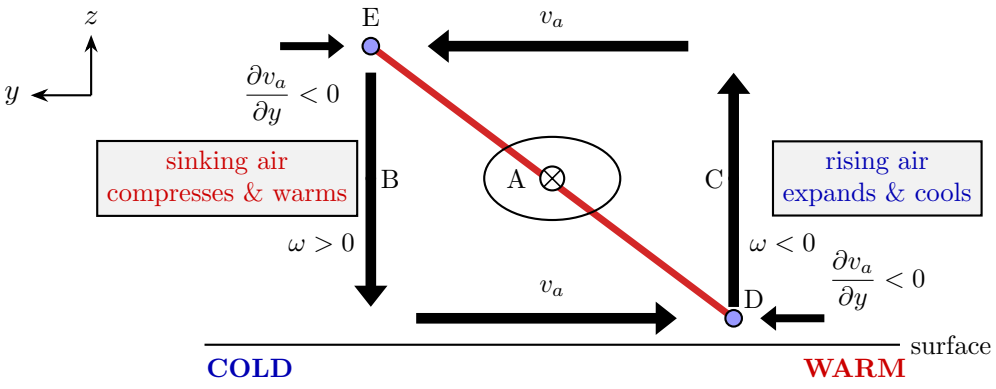
(The last term, the tilting of the perturbation temperature field by the vertical motion, neglected here.) For a front with $\partial T/\partial y < 0$ (cold to the north of warm), a **convergent** cross-frontal ageostrophic flow ($\partial v_a/\partial y < 0$) gives

$$-\underbrace{\frac{\partial v_a}{\partial y}}_{<0}\underbrace{\frac{\partial T}{\partial y}}_{<0} < 0 \qquad \Longrightarrow \qquad \frac{D}{Dt}\left(\frac{\partial T}{\partial y}\right) < 0$$

i.e., it leads to **frontogenesis**! Where does the ageostrophic flow converge? From mass continuity,

$$\frac{\partial v_a}{\partial y} + \frac{\partial \omega}{\partial p} = 0 \qquad \Longrightarrow \qquad \frac{\partial v_a}{\partial y} = -\frac{\partial \omega}{\partial p}$$

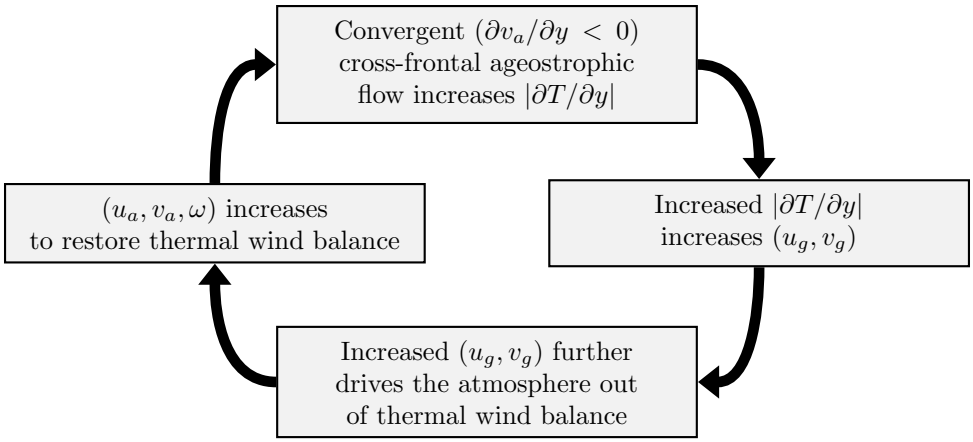
- At **Point D**, near the surface on the warm side, the air is rising above ($\omega < 0$), while $\omega = 0$ at the surface. Thus ω increases toward higher pressure, so that $\partial\omega/\partial p > 0$, and $\partial v_a/\partial y < 0$.
- At **Point E**, near the top of the circulation on the cold side, the air is sinking below ($\omega > 0$), while $\omega \approx 0$ above. Again $\partial\omega/\partial p > 0$, so that $\partial v_a/\partial y < 0$.



The advection of temperature by v_a leads to strong fronts at D and E!

A Positive Feedback

In QG, the ageostrophic circulation only acts to restore thermal wind balance. In SG, it *also* sharpens the front, which leads to a feedback loop:



This feedback loop leads to **very fast** rates of frontogenesis: in semi-geostrophic theory, the temperature gradient becomes **infinite** in around a day!

Properties of Semi-Geostrophic Frontogenesis

- A **very fast** rate of frontogenesis: an infinite temperature gradient forms in ≈ 1 day.
- The strongest temperature gradient (i.e., the front) **tilts with height toward the cold side**, along the line from D to E in the figure above.

These are both hallmarks of observed frontogenesis! For example, in the GFS cross-section across a cold front shown in class, there is sinking air on the cold side, rising air on the warm side, and a convergent cross-frontal ageostrophic flow in the lower troposphere (near 800 hPa). The zone of strongest frontogenesis slopes from roughly 750 hPa on the warm side up to roughly 550 hPa on the cold side, over a horizontal distance of only $\sim$200 km: exactly the structure that semi-geostrophy predicts. In classical schematics of a cold front, the warm, moist air is often depicted as "forced up" over the wedge of cold air, producing deep convective clouds (and often a dramatic shelf cloud) and precipitation along the front.

Frontogenesis Summary

- Horizontal (shearing and stretching) deformation by the **geostrophic** circulation concentrates temperature gradients, forming warm fronts ahead of and cold fronts behind the surface low.
- Across a front, the cross-front wind is **not** quasi-geostrophic (Ro ~ 1), so the advection by the ageostrophic wind cannot be neglected.
- **Semi-geostrophy** allows for advection by the **ageostrophic** wind. This turns out to be very important to model observed fronts:
  - it leads to much faster time scales of frontogenesis (≈ 1 day), and
  - it leads to fronts sloping with height toward the cold side.
- A strong feedback between the *geostrophic* circulation (which destroys thermal wind balance) and the *ageostrophic* circulation (which tries to restore thermal wind balance) leads to **frontogenesis**!